NICHOLAS  SAUNDERSON LL.D.

Lucasian Professor of Mathematicks in the University of Cambridge

Died 19. Ap. 1739. Aged 56.

I Vanderbank pinx. 1718. From the Original painted for Martin Folkes Esq. G. Vander Gucht Sculp.

THE

ELEMENTS of ALGEBRA,

IN TEN BOOKS:

(N. SAUNDERSON)

By NICHOLAS SAUNDERSON LL.D.

Late *Lucasian* Professor of the Mathematicks in the University of CAMBRIDGE, and Fellow of the Royal Society.

VOLUME THE FIRST:

Containing the Five first Books.

To which are prefixed

I. The LIFE and CHARACTER of the AUTHOR.

II. His PALPABLE ARITHMETIC Decyphered.

CAMBRIDGE,

Printed at the UNIVERSITY-PRESS:

MDCCXLI.



To the Right Honourable

JOHN Earl of RADNOR,

Viſcount BODMYN,

Baron ROBARTES of TRURO, &c.

MY LORD,

THE Relation I bore to the Author of the following Work, gives me a Right to intereſt myſelf in the Succeſs of it. To recommend it to the Publick with all poſſible Advantage, is the leaſt I owe to the Memory of the beſt of Fathers.

DEDICATION.

THE Reputation of an Author, and the Merit of a Performance, are in ordinary Cases the most effectual Recommendation to the favour of the Publick; and if I am not dec[illegible] by the Voice of Fame, nor flattered by the Report of Fr[illegible] the following Treatise wants neither of these Advantage[illegible]t whatever be the Case of the Author, or the Performa[illegible]e Subject itself, I am afraid, calls for some Countenance [illegible] Protection.

THE *Belles Le*[illegible]s, My LORD, lie open to all Mankind; every one that is capable of reading any thing, either has, or pretends to have some Relish for them. But Science lies more concealed, and more out of the reach of vulgar Apprehensions: Some have not Capacities or a just Taste for Mathematical Truths; and of those that have, few have Resolution enough to pursue them with a proper Industry: and be their Use and Importance ever so great, they will hardly emerge under such Disadvantages without some powerful Encouragement.

THESE, My LORD, are Difficulties arising from the nature of the Subject; to remove which, and to inspire the Youth with a generous Emulation for so noble a Science,

would

would be doing Service to the Publick, as well as Juſtice to the Author.

The Province of Exhortation and Inſtruction I leave to others, and content myſelf with the only Method that lies within my reach, of uſhering this Work into the World under the Authority and Protection of ſome great Name; one whoſe Birth and Quality may give a Dignity to Science, and by being a Maſter of it himſelf, has the juſter Title to patronize it in others; who does not meaſure the worth of a Performance by the fulſom Incenſe of the Dedicator, but penetrates into the Work itſelf, and learns from thence the real Abilities of the Writer.

It is for theſe Reaſons, My Lord, that I preſume to lay this Treatiſe at Your Feet. Suffer me to borrow Your Lordship's Name to throw a Luſtre upon my Subject, to attract the Attention of the Lazy and the Prejudiced; and to invite them by ſo great an Example to an Acquaintance with a Science, whoſe Uſe is almoſt as extenſive as Truth itſelf, and whoſe Invention does Honour to Human Reaſon.

Putting this Treatiſe under Your Lordship's Protection I look upon to be a Part of the Author's Will, and

which

which it is incumbent upon me to execute. Had He lived to give it to the Publick, I am confident he would have ſought no other Patron. Your LORDSHIP was his firſt Acquaintance in the Univerſity and his firſt Scholar, and, (what he ...er remembered with the ſincereſt Gratitude) his moſt g... Benefactor. A great Part of the Work was drawn up o... ...ly for Your LORDSHIP's Uſe, and was for many Years in ... ſole Poſſeſſion. And ſurely, My LORD, there is ſomethin... Juſtice in granting Protection to a Book, which firſt ... to You the Beauties of Mathematical Knowledge, and ...hich You are in ſome meaſure indebted for that excellent Taſt, that ſo eminently diſtinguiſhes Your LORDSHIP.

THE Author was honoured with the Friendſhip of moſt Perſons of Quality and Condition that ſtudied in the Univerſity, though it ended for the moſt Part with their Reſidence there. But Your LORDSHIP raiſes Friendſhips upon more laſting Principles; the Moment a Man of Merit knows You, he becomes your Friend, and when he is once ſo, he can never be otherwiſe. The Author had a Heart extremely ſenſible to true Worth; it received a very ſtrong and durable Impreſſion from the uncommon Share of it he obſerved in Your LORDSHIP; his Eſteem purſued You into the

World,

World, and continued undiminiſhed 'till he left it; he loved and honoured You ſincerely and paſſionately when living, and mentioned You with peculiar Tenderneſs and Affection in his dying Moments.

I CANNOT recal with Indifference a Scene in which I bore ſo conſiderable a Part, though my Youth rendered me incapable of taking my juſt Share of it. He was ſnatched away at a time when I was unable to make a true Eſtimate of the Loſs; when I had only experienced the Tenderneſs of the Parent, without enjoying the more ſolid Benefit of the Friend and the Inſtructor.

BUT I will ſhut up a Subject, which can give Your LORDSHIP no Pleaſure, and which muſt give me the tendereſt Concern. It is an Event that had been truly fatal to me, had not Your LORDSHIP's Goodneſs prevented it. This has taught me to bear a Misfortune which is the common Lot of all Men; and 'tis from this I am daily learning (as far as the Ties of Nature and filial Piety will permit) to forget it. Suffer me, My LORD, to boaſt (for there is ſomething of Vanity as well as Gratitude in it) that in Your ſingular Generoſity and Indulgence I have found the ſubſtantial Comfort of the Friend and the Parent. It is with Pride and

Pleaſure

Pleasure that I proclaim to the World my Obligations to You, and it is with the warmest Gratitude, and the most perfect Regard that I profess myself,

My LORD,

Your LORDSHIP'S

most obliged and

devoted Servant

JOHN SAUNDERSON.

THE

ELEMENTS

OF

ALGEBRA,



IN TEN BOOKS:

By NICHOLAS SAUNDERSON *LL.D.*

Late *Lucasian* Profeſſor of the Mathematics in the Univerſity of CAMBRIDGE, and Fellow of the Royal Society.

To which is prefixed,

An Account of the AUTHOR's LIFE and CHARACTER,

Collected from his oldeſt and moſt intimate ACQUAINTANCE.

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ADVERTISEMENT.

THE learned Author of this Book having, in his paper of Proposals, drawn up a short Scheme of his Work, it is thought proper to reprint it here, in order to give the Reader in a few words an idea of his performance.

This Work (*says the Author*) is chiefly intended for the instruction of young beginners, and for the use of those who have such under their care. It is divided into ten books, with an introduction prefixed concerning vulgar and decimal fractions; the former being absolutely necessary to a thorough knowledge of Algebra. In this introduction, all the reductions and operations of fractions are fully and clearly accounted for, and such a fund of reasoning established thereby in the mind of the learner, as cannot fail to furnish him with proper rules to work by in all cases wherever fractions are concerned, without any further assistance.

The first book treats of the nature of Algebra and Algebraic quantities; of their addition, subtraction, multiplication and division; of proportion, fractions, and extraction of roots in Algebra; and in the last place, of the manner of resolving simple equations, illustrated by a considerable number of examples. In this book, under the head of multiplication is shewn, how by bare multiplication alone many useful theorems may be invented, and have been invented, both in Arithmetic and Geometry. Under the heads of division and extraction of roots, some account is given of the rise and continuation of infinite serieses, and how they may be tried by involution; but they are here touched upon only so far as may be apprehended by the meanest capacities, all other considerations concerning them being reserved to other parts of this treatise, where it may reasonably be presumed, the learner will be better prepared for them.

The second book contains a great variety of entertaining questions producing simple equations, and solved partly by single positions, and partly by more; where all the useful methods of extermination are explained.

The third book treats of quadratic equations, and of the manner of resolving them, exemplified in various questions introduced for that purpose; where throughout the whole are occasionally interspersed many curious observations concerning the roots of equations, both possible and impossible.

The fourth book treats of pure Algebra, that is, where letters of the alphabet are used, not only to represent unknown quantities, but also such as are known, which in tenderness to the learner has hitherto been avoided. Here several of the former problems are resumed, more indefinitely

finitely proposed, and general solutions given them by general theorems or canons, first traced out analytically, and afterwards demonstrated synthetically, whereby the learner may make himself thorough master of both sorts of demonstrations.

The fifth book gives the solutions of many curious questions of that sort which admit of more answers than one, and some even of an infinite number, by a general method very easy to be comprehended, and (as the author conceives) entirely new. Here are demonstrated many elegant and useful theorems relating both to whole numbers and fractions; particularly that of Mr. *Cotes*, for finding the least numbers that will express a given ratio to any given degree of exactness. Here also occasion is taken to introduce the reader into an acquaintance with the most entertaining parts of *Euclid*'s doctrine of Incommensurables, where he will meet with more subtil and more refined reasoning than perhaps in any other part of the Mathematics whatever; and all levelled to the lowest capacity.

The sixth book is a choice collection of such questions as are usually known by the name of *Diophantine questions*; the solutions of them here given are very easy and intelligible, and in some cases perhaps may be thought preferable to any solutions of the same problems in *Diophantus*, or in any of his commentators or followers: for as *Diophantus*'s porisms are entirely lost, he lies, in many cases, at the mercy of his commentators.

The seventh book treats in the first place of the doctrine of proportion as it is delivered in the fifth book of the Elements. Here it is shewn that the common idea of proportionality may, without the least disguise, be so enlarged, as to extend to Incommensurables: and having thus established an infallible and adequate criterion of proportionality, the fifth and seventh definitions to the fifth book of the Elements are shewn to be no more than plain and natural consequences of it, but more proper upon many accounts for *Euclid*'s purpose, in carrying on his system of Geometry. Then all the propositions of the fifth book of the elements are clearly and succinctly demonstrated in their order, and as near as possible, in *Euclid*'s manner, so as to lose nothing either of the force or elegance of his demonstrations; and yet such an easy and familiar turn is given to the whole, that it is to be hoped this book, which has always been looked upon as an unsurmountable rub in the way through the six first books of the Elements, will now be read with as much ease to the imagination, nay more, than any other part of the Elements whatever. The latter part of this book gives a clear and distinct account of the composition and resolution of ratios, and of their great use in Natural and Mechanical Philosophy: insomuch that it is to be hoped, this part of the doctrine of proportion will be no longer a mystery to any one who will read it with the least degree of attention.

The

The eighth book applies Algebra to Geometry, and by the help of a few plain and easy problems, conveys to the mind of the learner the most sublime mysteries of that science. Here the composition of the Geometrical problems is first deduced from the *analysis*, and then synthetical demonstrations are formed from the constructions there given, without any regard to the *analysis*. The second part of this book contains the doctrine of solids, so far as it relates to prisms, cylinders, pyramids, cones, spheres, spheroids, &c; the principal properties whereof are taken out of *Euclid*, *Archimedes* and others, and demonstrated after the simplest manner, without the least stress laid upon the imagination.

The ninth book consists of several miscellaneous tracts; as first, of powers and their indexes; 2dly, of *Newton*'s method of evolving a binomial, considered in it's full extent; 3dly, of logarithms, their nature and use, and particularly of *Briggs*'s logarithms; 4thly, of logarithmotechny, or the method of computing logarithms, drawn from their simplest properties; 5thly, of *Newton*'s invention of divisors; and 6thly, of the Arithmetic of surd quantities.

The tenth and last book treats first of equations in general, and then of cubic and biquadratic equations in particular, gives the best methods of resolving them where they will admit of an accurate resolution, and proper rules for approximations where they will not; particularly *Newton*'s method is here described and explained.

By the account here given it is easy to see, that the author intended this treatise, not as a course of Algebra only, but also to promote, as far as possible, the study of Geometry, by removing or explaining all those difficulties which, from a long experience, he knows are apt to retard, if not discourage, young students in their progress through the Elements.

Though great care hath been taken to give the publick a correct edition of this excellent work, yet there are still remaining several errors of the press, though most of them are of little consequence. If the candid reader will take the trouble to correct the most material of those that are mentioned in the catalogue of errata, *I hope and believe, he will find no others that do in the least obscure the sense of the Author.*

There is one passage in the ninth book, relating to some matters of fact, wherein I conceive, the learned Author was mistaken as to some few particulars; concerning which I had not an opportunity to discourse with him, because of his sudden and unexpected sickness and death. The passage is in page 620, where the Author having recommended Mr. Briggs's *system of logarithms as the best accommodated for practice which are now in use, proceeds in the following words:* The Lord *Napeir*, a Scotch Nobleman, was the first inventer of logarithms: but our countreyman Mr. *Briggs*, Professor of Geometry in *Gresham* College, was undoubtedly the first who thought of

this

this system, and proposing it to the noble inventer, the Lord *Napeir*, he afterwards published it with that Lord's consent and approbation.

Now in the first *place I observe, that our late worthy Professor wrote the name of the celebrated inventer of logarithms differently in different parts of his manuscript copy. Sometimes he is called* Nepier, *and sometimes* Neper. *I have an english book written by this illustrious author, and printed in the year* 1611, *which in the title page is said to be* set foorth by *John Napeir* L. of Marchiston: *accordingly I have every where in the following work, caused his name to be printed* Napeir.

Secondly, *whereas our author stiles him* The Lord *Napeir*, a Scotch Nobleman; *this, I think, ought to be understood in a qualified sense. In* The Peerage of Scotland *written by* George Crawfurd *Esq*; *and printed at* Edinburgh *in* 1716, *I find, page* 364, *that Sir* Archibald Napier, *eldest son to the foresaid* John, *was the first of this family that was created a Peer of* Scotland, *being raised to that honour by King* Charles *the first in the year* 1627, *his father (the inventer of logarithms) having departed this life several years before, in the reign of King* James *the sixth. I confess, Mr.* Briggs *writes him* Johannes Neperus Baro Merchistonii: *but if the account given of this family by Mr.* Crawfurd *be true, the inventer of logarithms can only be supposed to have been one of the inferior order of Barons in* Scotland, *and not a Peer of that Realm.*

Thirdly, *whereas Dr.* Saunderson *stiles Mr.* Briggs, Professor of Geometry in *Gresham* College; *it is true, he was so, when* Napeir *first published his* Canon mirificus logarithmorum: *but when Mr.* Briggs *published his own* Arithmetica logarithmica *in* 1624, *he was* Savilian *Professor of Geometry at* Oxford, *as appears from the title page of that book.*

Fourthly, *whereas Mr. Professor* Saunderson *saith, that* our countreyman *Briggs* was undoubtedly the first who thought upon this system of logarithms which is now in use; *I know not by what authority he asserts this. On the contrary, Dr.* Keill *in the preface to his treatise* de logarithmis *writes thus:* Aliam deinde magis commodam logarithmorum formam *Neperus* excogitavit, et communicato consilio cum Domino *Henrico Briggio*, Geometriae in Academia *Oxoniensi* Professore, hunc socium operis sibi adjunxit, ut logarithmos in meliorem formam redactos compleret. *See likewise* Briggs's *preface to his* Arithmetica logarithmica.

I have nothing further to add, but that the Reader is obliged to the ingenious Mr. Abraham de Moivre *for two excellent performances inserted in this work: the former at the end of the fourth book, is his curious solution of two problems concerning proportionals; and the latter in an appendix to these Elements, is a noble discovery of a rule for extracting the cubick, or any other root of (what is called) an impossible binomial, such as* $a+\sqrt{-b}$, *and also for extracting any root out of a given power thereof.*

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ERRATA.

E R R A T A.

Note, The moſt material errors are marked with *.

* Pag 4. in the anſwer to the tenth queſtion, *for* 15625 feet *read* 15625 ſquare feet.
Pag. 45. lin. 8. *read* as any. And lin. 22, 23. *read* decimal.
Pag 57. lin. penult. *read* the two firſt products.
Pag. 64. lin. 6 from the bottom, *read* it was ſhewn.
* Pag. 138. lin. 12. *for* ſubſtitute *read* ſubſtituting.
* Pag. 239. lin. 4 from the bottom, for $s\times -q-qr=\overline{-qs-qrs}$ read $s\times \overline{-q-qr}=\overline{-qs-qrs}$.
* And in the laſt line, for $-\overline{q-q}r$ read $\overline{-q-qr}$.

In the Second Volume.

Pag 390. lin. 11. *dele* muſt.
* Pag. 536 lin. 14. for $x^2-a^2x^2$ read $x^4-a^2x^2$.
Pag. 551. lin. 21. *for* where *read* when.
Pag. 604. lin. ult. for x^{+3} read x^3.
Pag. 658. lin. 4. *read* DEFINITIONS.
* Pag. 671. lin. 3. *for* binomial *read* binomial root.
Pag 683. line 21. *read* left.
* Pag. 763. in the firſt line of the ſixth example, for $\frac{q}{4}$ read $\frac{qq}{4}$.
Pag 738. line 13. *dele* a.

In the Table of the Contents.

* Art. 231. for *two numbers* read *two ſquare numbers*.
Art. 403. for *Napeirs* read Napeir's.

Errata in the Pointing.

Pag. 44. lin. 16. *after* .4375 *put a comma.*
Pag. 81 lin. 18. at the end of the line, after $2aa$, inſtead of a comma put a ſemicolon.
Pag. 91. at the end of the fifth example, put a full ſtop inſtead of a comma.
Pag. 124. lin. 8 from the bottom, at the end of the line, put a ſemicolon inſtead of a comma.
Pag 167. lin. 25 *after* no ſquare number *put a ſemicolon inſtead of a comma.*
Pag. 178. line the laſt but one of the ſecond example, *after* ± 1 *put a ſemicolon.*
Pag. 199. lin. 16. after $x=\frac{32400-675x}{x}$ put a ſemicolon.
Pag 238. lin. 7. put a full ſtop at the end of the paragraph.
Pag. 331. lin. 5. at the end of the paragraph, put a full ſtop inſtead of a comma.

In the Second Volume.

Pag. 542. lin. 5 from the bottom, *for DEG*; but *read DEG*. But.
Pag. [illegible]. lin. 16. for *root: and* vice verſa read *root. And* vice verſa.
Pag. 738. lin. antepen. at the end of the line, after $xx+3x+2=0$ put a comma.

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WHEREAS ABIGAIL SAUNDERSON, Widow of the late NICHOLAS SAUNDERSON, Doctor of Laws, and Profeſſor of Mathematicks in His Majeſty's Univerſity of *Cambridge*, and JOHN SAUNDERSON, Son, and ANNE SAUNDERSON, Daughter of the ſaid NICHOLAS SAUNDERSON, have humbly repreſented unto Us, that the ſaid NICHOLAS SAUNDERSON, did, in his Life-time, with great Labour and Expence compoſe a Treatiſe, Intituled, *The Elements of Algebra in Ten Books*, now printing at *Cambridge* in Two Volumes *in Quarto*, with the *Memoirs of the Author's Life*, the Property of which Treatiſe and Memoirs is in the abovementioned ABIGAIL, JOHN and ANNE SAUNDERSON; and have farther repreſented unto Us, that the ſaid NICHOLAS SAUNDERSON, did alſo compoſe ſeveral other Mathematical and Philoſophical Treatiſes, the Property of which laſt mentioned Treatiſes, is in the aforeſaid JOHN SAUNDERSON; and have therefore humbly beſought Us to grant unto them the ſaid ABIGAIL, JOHN and ANNE SAUNDERSON, His Majeſty's Royal Privilege and Licenſe, for the ſole printing and publiſhing the aforeſaid Treatiſe and Memoirs, Intituled, *The Elements of Algebra in Ten Books*, with the *Memoirs of the Author's Life*, for the Term of Fourteen Years, and alſo unto the ſaid JOHN SAUNDERSON His Majeſty's Royal Privilege and Licence for the ſole printing and publiſhing the reſt of the Treatiſes abovementioned, for the ſaid Term of Fourteen Years; We, being willing to give due Encouragement to Works of this Nature, that tend to the Advancement and Propagation of Learning, have thought fit to condeſcend to their Requeſt, and do therefore in His Majeſty's Name, by theſe Preſents, ſo far as may be agreeable to the Statutes in that behalf made and provided, grant unto them the ſaid ABIGAIL, JOHN, and ANNE SAUNDERSON, their Heirs, Executors and Aſſigns, His Majeſty's Royal Privilege and Licence, for the ſole printing and publiſhing the aforeſaid Treatiſe and Memoirs, Intituled, *The Elements of Algebra in Ten Books*, with the *Memoirs of the Author's Life* for the Term of Fourteen Years; and alſo unto the ſaid JOHN SAUNDERSON, his Heirs, Executors and Aſſigns for the ſole printing and publiſhing the reſt of the Treatiſes abovementioned for the ſaid Term of Fourteen Years, to be computed from the Date hereof; ſtrictly forbidding and prohibiting all His Majeſty's Subjects, within His Majeſty's Kingdoms and Dominions to reprint, abridge, or tranſlate any of the abovementioned Treatiſes, either in the like, or any other Volume or Volumes whatſoever, or to import, buy, vend, utter or diſtribute any Copies of them, reprinted beyond the Seas, during the ſaid Term of Fourteen Years, without the Conſent and Approbation of the ſaid ABIGAIL, JOHN and ANNE SAUNDERSON, their Heirs, Executors and Aſſigns, by Writing under their Hands and Seals firſt had and obtained, as they and every of them Offending herein, will anſwer the contrary at their Perils; Whereof the Commiſſioners and other Officers of His Majeſty's Cuſtoms, the Maſter, Wardens and Company of Stationers of the City of *London*, and all other His Majeſty's Officers and Miniſters, whom it may concern, are to take notice, that due Obedience be given to theſe Preſents.

Given at *Whitehall* the Twenty Ninth Day of *May*, One Thouſand Seven Hundred and Forty, and in the Thirteenth Year of His Majeſty's Reign.

By their EXCELLENCYS *Command*

ANDREW STONE.

MEMOIRS

OF THE

LIFE and CHARACTER

OF

Dr. *Nicholas Saunderſon*,

Late *Lucaſian* Profeſſor of the Mathematics in the Univerſity of CAMBRIDGE.

THE following Treatiſe was entirely finiſhed, before the Public was engaged in the Subſcription; but the Author dying during the Impreſſion of the Work, it is believed ſome ſhort Account of a Perſon ſo remarkable and ſingular will be no unacceptable Entertainment to the Reader. Therefore the following Gentlemen, Dr. THOMAS NETTLETON of *Halifax*, Dr. RICHARD WILKES of *Woolverhampton*, the Rev. Mr. JOHN BOLDERO late Fellow of *Chriſt's* College, the Rev. Mr. GERVAS HOLMES late Fellow of *Emmanuel* College, the Rev. Mr. GRANVILE WHEELER, and Dr. RICHARD DAVIES of *Shrewsbury* late Fellow of *Queen's* College, who were the intimate Friends of the Deceaſed, in the different Parts of his Life from his Youth to the time of his Death, have communicated the following Particulars.

Mr. NICHOLAS SAUNDERSON was born in *January* 1682, at *Thurlston* near *Penniston* in *Yorkshire*. His Father, besides a small Estate, had a Place in the Excise, which he enjoyed above Forty Years with good Repute. His eldest Son, of whom we are speaking, when Twelve Months old, was deprived by the Small Pox, not only of his Sight, but his Eyes also, for they came away in Abscess. A Sense so little enjoyed was soon forgot; he retained no more Idea of Light and Colours than if he had been born Blind.

HE was sent early to the Free School at *Penniston*, and under the Instruction of Mr. STANIFORTH, laid the Foundation of that Knowledge of the Greek and Roman Languages, which he afterwards improved so far by his own Application to the Classic Authors, as to hear the Works of *Euclid*, *Archimedes* and *Diophantus* read in their original Greek: *Virgil* and *Horace* were his Favourites among the Roman Poets: His Memory was well stored with their most beautiful Passages, and he would frequently in Conversation quote them with great Propriety. He was well versed in the Writings of *Tully*, and dictated Latin in a familiar and elegant Style. He afterwards acquired a competent knowledge of the French Tongue.

As soon as he had gone through the Business of the Grammar School, his Father, whose Occupation led him to be conversant in Numbers, began to instruct him in the common Rules of Arithmetic. Here it was his Genius first appeared; he soon became able to work the common Questions, to make long Calculations by the strength of his Memory, and to form new Rules to himself for the more ready solving of such Problems, as are often proposed to Learners, more with a design to perplex than instruct: so that in all Difficulties, his Schoolfellows generally applied to him instead of their Master.

At the Age of Eighteen he was introduced to the acquaintance of RICHARD WEST of *Underbank* Esq; a Gentleman of Fortune and a Lover of the Mathematics: who observing Mr. SAUNDERSON'S uncommon Capacity, took the Pains to instruct him in the Principles of Algebra and Geometry, and gave him every Encouragement in his Power to the Prosecution of these Studies: foreseeing of what Advantage to Letters so great a Genius might be. Soon after, he grew acquainted with Dr. NETTLETON; and it was to the great Pleasure these Gentlemen took in assisting and improving him in his Studies, that our Author owed his first Institution in the Mathematic Sciences. They furnished him with Books, and often read and expounded them to him: but he soon surpassed his Masters, and became fitter to teach, than learn any thing from them.

OUR Author's Passion for Learning grew with him, and his Father, willing to encourage this laudable Disposition, sent him to a private Academy at *Attercliff* near *Sheffield*. Logic and Metaphysics made up the principal Learning of this School. The former being chiefly the Art of Disputing in Mood and Figure, a dry Study, much conversant in Words, the latter dealing in such abstract Ideas as have not the Objects of Sense for their Foundation, were neither of them agreeable to the Genius of our Author; he therefore made but a short Stay here for Instruction.

AFTER he left this Place, he remained some time in the Country, prosecuting his Studies in his own way, without any Guide or Assistant: indeed he needed no other than a good Author, and some Person that could read it to him: by the Strength of his own Genius he could easily master any Difficulty that occurred therein. His Education had hitherto been carried on at the Expence of his Father, who having a numerous Family, grew uneasy under the Burden. His

Friends therefore began to think of fixing him in ſome way of Buſineſs, by which he might ſupport himſelf. His Inclination led him ſtrongly to the Univerſity of CAMBRIDGE, where he expected to meet with the beſt Opportunities of Improvement in his favourite Studies. But the great Expence of Education, the length of Time he muſt continue there to obtain his Degrees, and be duly qualified for any of the Liberal Profeſſions, were Difficulties not to be ſurmounted. At laſt it was reſolved he ſhould try his Fortune there, but in a way very uncommon; not as a Scholar but a Maſter: for his Friends, obſerving the extraordinary Proficiency which he had already made in Mathematical Learning, and a peculiar Felicity of Expreſſion in conveying his Ideas to others, were ſanguine in their Hopes, that he might teach the Mathematics with Credit and Advantage, even in the Univerſity. Or if this Deſign ſhould miſcarry, they promiſed themſelves Succeſs in opening a School for him in *London*.

Accordingly in the Year 1707, being now Twenty-five Years of Age, he was brought to CAMBRIDGE by Mr. JOSHUA DUNN, then a Fellow-Commoner of *Chriſt*'s College, where he reſided with his Friend, but was not admitted a Member of the College. The Society were extreamly pleaſed with ſo unuſual a Gueſt, allotted him a Chamber, the uſe of their Library, and indulged him in every Privilege that could be of Advantage to him. But many Difficulties obſtructed his Deſign: He was placed here without Friends, without Fortune, a Youth untaught himſelf, to be a Teacher of Philoſophy in an Univerſity where it then reigned in the greateſt Perfection. Mr. WHISTON was at this time in the Mathematical Profeſſor's Chair, and read Lectures in the manner propoſed by Mr. SAUNDERSON; ſo that an Attempt of this kind looked like an Encroachment on the Privileges of his Office. But as a good-natured Man and an Encourager of

Learning

Learning, he readily consented to the Application of Friends made in behalf of so extraordinary a Person. Mr. DUNN had been very assiduous in making known his Character; his Fame in a few Months had filled the University, so that Men of Learning and Curiosity grew ambitious and fond of his Acquaintance. His Lecture, as soon as opened, was attended by many from several of the Colleges, and in some time was so crowded, that he could hardly divide the Day among all who were desirous of his Instruction: few, whose Inclination led them to the severer Studies, but eagerly embraced the Opportunity of laying a Foundation in Mathematics and Philosophy under so great a Master.

Sir ISAAC NEWTON had left *Cambridge* several Years before Mr. SAUNDERSON came thither. His *Principia Mathematica* had been long since published, but were at first overlooked, and not sufficiently understood by the World. It was one Design of this Treatise to demolish the *Vortices* and other romantic Chimæras of DES CARTES: and now the Learned began to be sensible how much the Author had done towards a Reformation in Philosophy; which before had been founded upon very erroneous Principles, and *Hypotheses* feigned in the Closet, without one Experiment to shew their Reality in Nature. Sir ISAAC, ever studious of brevity, had drawn up his Demonstrations in the concisest manner possible; leaving the Mathematical Reader to furnish himself with every thing before known, and often to take large Steps alone. His Treatise of *Optics* and his *Arithmetica Universalis* were both written in the same masterly Style, and each contain great and peculiar Discoveries. Mr. SAUNDERSON made these several Pieces the Foundation of his Lecture; they afforded a noble Field to display his Genius in; and the Public Schools of the University did sufficiently testify his Success. For those wonderful *Phænomena* of Nature, whose Solution

lution was before attained with Difficulty by the beſt Mathematicians, became the *Theſes*, which the Youth of three or four Years ſtanding defended in their Diſputations for their firſt Degree in Arts. We every Year heard the Theory of the Tydes, the *Phænomena* of the Rainbow, the Motions of the whole Planetary Syſtem as upheld by Gravity, very well defended by ſuch as had profited by his Lectures.

IT will be matter of ſurpriſe to many that our Author ſhould read Lectures in Optics, diſcourſe on the Nature of Light and Colours, explain the Theory of Viſion, the Effect of Glaſſes, the *Phænomena* of the Rainbow, and other Objects of Sight: but if we conſider that this Science is altogether to be explained by Lines, and ſubject to the Rules of Geometry, it will be eaſy to conceive that he might be a Maſter of theſe Subjects.

AS Mr. SAUNDERSON was inſtructing the Univerſity Youth in the Principles of *Newtonian* Philoſophy, it was not long before he became acquainted with the incomparable Author, and enjoy'd his frequent Converſation concerning the more difficult Parts of his Works. Dr. HALLEY, Mr. DE MOIVRE and many of the moſt noted Mathematicians in *London* highly eſteemed his Friendſhip, and in deference to his ſtrong Reaſon and Judgment, frequently conſulted him concerning their Writings and Deſigns.

UPON the removal of Mr. WHISTON from his Profeſſorſhip, Mr. SAUNDERSON's Mathematical Merit was univerſally allowed ſo much ſuperior to that of any Competitor in the Univerſity, that an extraordinary Step was taken in his Favour, to qualify him with a Degree, which the Statutes require. Upon Application made by the Heads of Colleges to the Duke of SOMERSET their Chancellor, together with the Interceſſion of the Honourable FRANCIS ROBARTES Eſq; a Mandate was readily granted by the QUEEN, for conferring on him the Degree

gree of Maſter of Arts. Upon which he was choſen *Lucaſian* Profeſſor of the Mathematics in *November* 1711. During this whole Tranſaction Sir ISAAC NEWTON intereſted himſelf very much in his Favour.

AT this time the ingenious Mr. ROGER COTES filled the *Plumian* Chair of Aſtronomy and Experimental Philoſophy; a Man of great ſweetneſs of Temper, and engaged to our Author in the ſtricteſt Friendſhip; of the ſame Age, of the ſame Genius and Inclination to the Mathematics, both approved and recommended to Profeſſorſhips by Sir ISAAC NEWTON. No Univerſity could ever at one time boaſt of two ſo capable and ſo diſpoſed to promote the Study of Philoſophy among her Pupils. Had they lived to more mature Ages, mutually aſſiſting and inſpiring each other in the purſuit of Knowledge, what Glory might have accrued to our Univerſity, what Advancement to Science from their united Labours! But Mr. COTES was hurried away by a Fever in the Flower of his Age, having only time to compoſe a few Pieces, as Specimens of his extraordinary Capacity, but of great value to the Learned. And our Author's Life, though longer, was ſo devoted to Lectures, that he now leaves to Poſterity as few Monuments of his Abilities.

OUR Author's firſt Performance after he was ſeated in the Chair, was an Inauguration Speech, made in very elegant Latin, and a Style truly *Ciceronian:* it was delivered with ſuch juſt Elocution, and in a manner ſo graceful, as to gain him the univerſal Applauſe of his Audience. In it he firſt returned his Thanks to Her MAJESTY for the Royal Mandate, to the Chancellor for his ready Application to the Queen, and to the Electors and the reſt of his Friends for their good Opinion of his Abilities and Mathematical Knowledge. To theſe he added a long and noble Encomium on the

the *Mathematics*, ſhewing the Excellence and Advantage of this above every other Method of Reaſoning.

FROM this time he applied himſelf cloſely to the reading of Lectures, and gave up his whole time to his Pupils: ſo that his Friends ſoon loſt all the Pleaſure of his Converſation. He continued among the Gentlemen of *Chriſt*'s College till the Year 1723, when he took a Houſe in *Cambridge*, and ſoon after married a Daughter of the Rev. Mr. WILLIAM DICKONS late Rector of *Boxworth* in the County of *Cambridge*; by whom he had a Son and a Daughter, both now living.

IN the Year 1728, when His preſent Majeſty King GEORGE the Second honoured the Univerſity of *Cambridge* with a Royal Viſit, He was pleaſed to ſignify his deſire of ſeeing ſo remarkable a Perſon. Accordingly our Profeſſor attended upon His Majeſty in the Senate Houſe, and was there created Doctor of Laws by his Royal Favour.

Dr. SAUNDERSON was naturally of a ſtrong, healthy Conſtitution; but being too ſedentary, and conſtantly confining himſelf to his Houſe, he became at length a Valetudinarian of a very Scorbutic Habit. For ſome Years he frequently complained of a Numbneſs in his Limbs, which in the Spring of the Year 1739 ended in a Mortification in his Foot. His Blood was in ſo ill a State that no Art or Medicines were able to ſtop its Progreſs. He died the 19th of *April* 1739; in the Fifty-ſeventh Year of his Age, and lies buried according to his laſt Requeſt in the Chancel at *Boxworth*.

AFTER his Life, it may be expected that ſome Account ſhall be given of his Character likewiſe; but I am at a loſs for Colours, ſtrong enough to paint a Character ſo bright and uncommon, and where to place it for View, in the trueſt Point of Light. A blind Man moving in the Sphere of a Mathema-

Mathematician ſeems a *Phænomenon* difficult to be accounted for; and has excited the Admiration of every Age in which it has appeared. *Tully* mentions it as a thing ſcarce credible in his own Maſter in Philoſophy *Diodotus*[a], "that "he exerciſed himſelf therein with more aſſiduity, after he "became blind: and what he thought next to impoſſible "to be done without Sight, that he profeſſed Geometry, de- "ſcribing his Diagrams ſo expreſsly to his Scholars, that they "could draw every Line in its proper Direction." St. *Jerom* relates a more remarkable Inſtance in *Didymus*[b] of *Alexandria*, who "though blind from his Infancy, and there- "fore ignorant of the very Letters, appeared ſo great a Mi- "racle to the World, as not only to learn Logic, but Geo- "metry alſo to perfection, which ſeems the moſt of any thing "to require the help of Sight." The Character of *Didymus* is celebrated, among other Hiſtorians, by *Caſſiodorus*; who makes mention alſo of one *Euſebius*[c] an *Aſiatic*, who according to his own Account of himſelf, "had been blind from "five Years old, and yet had treaſured up in his Mind all "kinds of Learning, and explained them likewiſe with the "greateſt clearneſs to others." And *Trithemius* gives a like Inſtance in one "*Nicaiſe*[d] of *Mechlin*, who though blind from

[a] Cic. Tuſc. Diſp. V, 39. *Diodotus Stoicus, cæcus multos annos, noſtræ domi vixit: is verò, quod credibile vix eſſet, cùm in Philoſophia multò etiam magis aſſiduè quam antea verſaretur, ——— tum quod ſine oculis fieri poſſe vix videtur, Geometriæ munus tuebatur, præcipiens diſcentibus, unde, quo, quamque lineam ſcriberent.*

[b] Hieronymus de viris Illuſt. Cap. CIX. *Didymus Alexandrinus captus a parva ætate oculis, & ob id elementorum quoque ignarus, tantum miraculum ſui omnibus præbuit, ut Dialecticam quoque & Geometriam, quæ vel maximè viſu indiget, uſque ad perfectum didicerit.*

[c] Caſſiodorus de Inſt. Div. Liter. cap. 5. tradit *de partibus Aſiæ quendam ad nos* veniſſe *Euſebium nomine, qui ſe infantem quinque annorum ſic cæcatum eſſe narrabat, ut ſiniſtrum ejus oculum fuiſſe excavatum orbis profundiſſimus indicaret: dexter verò globus vitreo colore confuſus ſine videndi gratia infructuoſis niſibus volvebatur. Hic —— diſciplinas omnes & animo retinebat, & expoſitione planiſſima lucidabat.*

[d] Trithemius de Scriptoribus Eccleſ. N. DCCCLXXVI. *Nicaſius de Voerda, Mechlinienſis, ——— captus à tertio ætatis ſuæ anno oculis, ——— ſecundum noſtra ætate Didymum Alexandrinum exhibuit, dum in omni doctrina & ſcientia, tam divina quam humana eruditiſſimus evaſit. Nam in gymnaſio Colonienſi ——— jura publicè docuit, libros utriuſque juris, quos nunquam vidit, auditu didicit, tenuit mente, apertè recitavit.*

" the third Year of his Age, yet, like another *Didymus*, be-
" came ſo great a Maſter of all Learning and Knowledge,
" divine and human, that in the Univerſity of *Cologne*, he
" publicly taught the Canon and Civil Law, openly reciting
" Books which he had never ſeen, but had learnt by only
" hearing them read to him." Beſide theſe I have heard mention made of an *Hollander*, if not ſome others, who notwithſtanding their Blindneſs have excelled in Mathematical Learning.

It is remarkable of the few, who have laboured under this Defect, and the ſtill fewer, who had Genius enough to ſurmount the Difficulties attending it, that ſo many ſhould be found to excel in Learning, and particularly in the Mathematics, as the two firſt above mentioned *certainly* did, and *probably* the others alſo. But if we conſider that the Ideas of extended Quantity, which are the chief Objects of Mathematics, may as well be acquired from the Senſe of Feeling as that of Sight; that a fixed and ſteady Attention is the principal Qualification for this Study, and that the Blind are by neceſſity more abſtracted than others, we ſhall perhaps find Reaſon to think there is no other Branch of Science more adapted to their Circumſtances. It is ſaid of *Democritus* that he put out his Eyes, to enable him to think the more intenſely; " imagining, ſays *Tully*[e], the Acuteneſs " of the Mind was taken off by the Sight of the Eye." And it was an Obſervation frequently made by our Profeſſor, that Diagrams which are intended only as helps to the Imagination, are often the means of miſleading the Judgment. It is certain, however uſeful they may be to the Learner, yet the Inventer muſt in all Caſes proceed without them. The Scheme muſt be erected in his Imagination, in Circumſtances as ge-

[e] Cic. Tuſc. Diſp. V, 39. *Democritus —— impediri etiam animi aciem aſpectu oculorum arbitrabatur.*

neral

neral as the Propoſition, ſuch as cannot be delineated upon Paper. And I am confident, that any one who is deſirous of more than a general Knowledge of theſe Things, who would invent and improve upon what is to be learned from Books, will find his Mind greatly aſſiſted and enlarged, by accuſtoming himſelf to think and reaſon in the Circumſtances of a blind Man. But a Perſon who has the Misfortune to be ſuch, and is deprived of all the Pleaſures of Sight, will more frequently and more cloſely retire into himſelf, and finding few other Amuſements but in the purſuit of Truth, will be more likely to excel in theſe abſtract Sciences.

THE ſame Circumſtance may poſſibly contribute ſomething towards raiſing the Genius beyond its natural Pitch, in ſome other Arts, particularly Muſic and Poetry. The Poet indeed muſt firſt have his Imagination filled with all the beautiful Variety of Images in Art and Nature, which the Sight only can ſupply: if he then be deprived of that Senſe,

So much the rather may Celeſtial Light
Shine inward, and the Mind through all her Powers
Irradiate: —

as our blind Poet expreſſes it. Accordingly in the Catalogue of Epic (the ſublimeſt kind of) Poets, we find two *blind Bards*, ſurpaſſing all that any Age or any Nation have produced in the Flights of Fancy. And I cannot but wonder the very ingenious *Enquirer* into the Life and Writings of *Homer*, who endeavours to account for the great Genius, from a Concurrence of natural Cauſes, ſhould take no notice of that Circumſtance, which was ſo peculiar to his Poet.

IT was by the Senſe of Feeling our Author acquired moſt of his Ideas at firſt: and this he enjoyed in great Acuteneſs and Perfection, as it commonly happens to the Blind, whether by the kind Gift of Nature, or the Neceſſity of Applica-

tion. Yet he could not, as ſome have imagined, (and as Mr. *Boyle* was made to believe of a blind Man at *Maeſtricht*) diſtinguiſh Colours by that Senſe; and having made repeated Tryals himſelf, he uſed to ſay, it was pretending to Impoſſibilities. But he could with great Nicety and Exactneſs diſcern the leaſt Difference of Rough and Smooth in a Surface, or the leaſt Defect of Poliſh. Thus he diſtinguiſhed in a Set of *Roman* Medals, the genuine from the falſe, though they had been counterfeited with ſuch Exactneſs as to deceive a *Connoiſſeur*, who had judged by the Eye. But ſays the Profeſſor, "I, who had not that Senſe to truſt to, could "eaſily feel a Roughneſs in the new-caſt, ſufficient to diſtin-"guiſh them by." His Senſe of Feeling was very accurate in diſtinguiſhing the leaſt Variation in the Atmoſphere. I have been preſent with him in a Garden, making Obſervations on the Sun, when he has taken notice of every Cloud that diſturbed our Obſervation, almoſt as juſtly as we could. He could tell when any thing was held near his Face, or when he paſſed by a Tree at no great Diſtance, provided the Air was calm, and little or no Wind: theſe he did by the different Pulſe of the Air upon his Face.

I wiſh I were capable of entertaining the Curious with the many Contrivances he had, to ſupply his Defect of Sight. He had a Board made with Holes bored at the equal Diſtance of half an Inch from each other: Pins were fixed in them, and by drawing a Piece of Twine round their Heads, he could more readily delineate all rectilinear Figures uſed in Geometry, than any Man could with a Pen. He had another Board with Holes made in right Lines for Pins of different Sizes. By the help of theſe he could calculate, and ſet down the Sums, Products, or Quotients in Numbers, as exactly as others could by Writing. By the help of an Armillary Sphere, the Schemes in Geometry that lie in different Planes,

Planes, and the regular Solids cut in Wood, and the Form of ſeveral Curves made after the ſame manner, he was able on theſe Subjects to convey the cleareſt Ideas to his Pupils.

A refined Ear is what ſuch are commonly bleſſed with, who are deprived of their Eyes. Our Profeſſor was perhaps inferior to none in the Excellence of his: he could readily diſtinguiſh to the fifth Part of a Note, and by his Performance on the Flute, which he had learned as an Amuſement in his younger Years, diſcovered ſuch a Genius for Muſic, as would probably have appeared as wonderful, as his Excellence in the Mathematics; had he cultivated that Art with equal Application. By his Quickneſs in this Senſe he not only diſtinguiſhed Perſons, with whom he had ever once converſed, ſo long as to fix in his Memory the Sound of their Voice, but in ſome Meaſure Places alſo. He could judge of the Size of a Room into which he was introduced, of the Diſtance he was from the Wall: and if ever he had walked over a Pavement in Courts, Piazzas, &c, which reflected a Sound, and was afterwards conducted thither again, he could exactly tell whereabouts in the Walk he was placed, merely by the Note it ſounded.

THE Reader muſt greatly admire the Strength of his Memory, when aſſured that he could calculate in his Mind, multiply, divide, extract the ſquare or cube Root to many Places of Figures; could go along with any Calculator in working Algebraical Problems, Infinite Series, &c; and immediately correct the Slips of the Pen, as well in Signs as in Numbers. Thoſe who read to him had frequent Occaſions of admiring his great Sagacity and Quickneſs of Conception; with how much eaſe he followed any Track of Reaſoning, and with what Art he ſtored up in his Mind ſuch Parts, as would ſerve him to recollect and ruminate upon the whole. Indeed in the more abſtruſe Parts of Mathematics, where the

Scheme

Scheme was very intricate and perplexed, they often found it difficult to raiſe in his Imagination a clear and diſtinct Perception of it: but that once done, he ſeldom or never required any farther Aſſiſtance; his Mind retained ſo ſtrongly every Impreſſion that was once rightly made upon it. By the help of theſe ſtrong Faculties, a clear Imagination, tenacious Memory, and quick Reaſon, the Books of Mathematics lay ever open to him; he ſaw the whole in one View, every Dependency in the Chain of Truth. Thus he knew how to found every thing on the moſt eaſy Principles, and to compoſe with the juſteſt Symmetry and Order.

As in the Knowledge of the Mathematics he was exquiſite, and equal to any, ſo in the Addreſs of a Teacher, he was perhaps ſuperior to all. This Quality was conſpicuous at his firſt Appearance in the World, and muſt have been highly improved by long Uſe and Experience. He ſeemed perfectly to know what Difficulties young Minds are apt to be involved in, and how beſt to obviate or remove them. His Expreſſion was ſtrong and clear; and his Method ſo juſt and natural, that no one was at a loſs to follow him. He was very happy in all the Arts of facilitating a Demonſtration, in forming curious Poſitions to help the Imagination and obviate the Difficulties of Conception. I dare appeal to the following Sheets to determine, whether the ſeveral Propoſitions, which have paſſed through the Hands of *Euclid*, *Archimedes*, *Diophantus*, and the greateſt Maſters both ancient and modern, have not been greatly improved under his, by lowering the Ground-Work, and rendering the Structure more plain, yet more uſeful and ſubſtantial.

His Inclination led him to thoſe Parts of the Mathematics, which are not the moſt abſtracted, and end only in Contemplation. A Propoſition muſt have its Uſes, in order to engage his Attention. Either the Method of Enquiry muſt help

help to form the Mind, and teach new Modes of Reaſoning, or the Propoſition itſelf muſt tend to ſome Good, to the Improvement of Life or Science. He conſidered Mathematics as the Key to Philoſophy, as the Clue to direct us through the ſecret Labyrinths of Nature; and thought the Mind was more highly entertained as well as improved in unravelling Her Works, than inveſtigating the moſt ſubtile Properties of abſtract Quantity.

As to the *Geometric* and *Analytic* Methods of Reaſoning, each of which have their Advocates and Favourers among the Mathematicians of the preſent Age: our Profeſſor, I think, did Juſtice to both, in allowing each the Advantage on different Occaſions, and making uſe of that which ſeemed the moſt proper for the preſent. The *Geometric* being the moſt Intuitive, and conveying the ſtrongeſt and cleareſt Ideas to the Mind, he allowed preferable, where equally obvious and eaſy of Application. But as it was often otherwiſe, the *Analytic* advancing us in Science much faſter and farther than we could have gone by all the Methods of the Ancients, and being the very Art and Principle of Invention, He thought the Moderns were greatly aſſiſted by the uſe of it.

Our Profeſſor would not be induced by the Deſires and Expectations of any, to engage in the War that was lately waged among Mathematicians, with no ſmall Degree of Heat, concerning the *Algorithm or Principles of Fluxions.* Yet he wanted not the greateſt Reſpect for the Memory of Sir Isaac Newton, and thought the whole Doctrine entirely defenſible by the ſtricteſt Rules of Geometry. He owned indeed that the great Inventer, never expecting to have it canvaſſed with ſo much trifling Subtilty and Cavil, had not thought it neceſſary to be guarded every where by Expreſſions ſo cautious as he might have otherwiſe uſed: for he wrote only for ſuch ſincere Lovers of Truth as himſelf was. But the general

Averſion

Averſion he had to all Controverſial Writings withheld him from appearing in this. However, as he intended another Volume to his *Algebra* on the *Fluxionary* Part, he there deſigned to be particularly accurate and explicit upon the *Algorithm*; with an indirect View to the Controverſy on foot, and to obviate, the beſt he could, every Difficulty that had been ſtarted.

I cannot upon this Occaſion paſs by the Name of Sir ISAAC NEWTON, without mentioning the profound Veneration paid to it by our Profeſſor. If he had ever differed in Sentiment from any of his Mathematical and Philoſophical Writings, upon more mature Conſideration, he ſaid, he always found the Miſtake to be his own. The more he read his Works, and obſerved upon Nature, the more Reaſon he found to admire the Juſtneſs and Care, as well as Happineſs of Expreſſion of that incomparable Philoſopher. He has left ſome valuable Comments on the *Principia*, which not only explain the more difficult Parts, but often improve upon the Doctrines, and which may in their preſent State be no unacceptable Preſent to the Public, though far ſhort of any thing he would himſelf have publiſhed on that Subject.

THERE was ſcarce any Part of the Mathematics, on which our Profeſſor had not wrote ſomething for the uſe of his Pupils. But he diſcovered no Intention of publiſhing any of his Works, till the Year 1733: when his Friends, alarmed by a violent Fever that had highly threatened his Life, and being unwilling that the Labours of ſo great a Man ſhould be loſt to the World, importuned him to ſpare ſome time from his Lectures, (which he then attended ſeven or eight Hours in a Day, to the great hazard and prejudice of his Health,) and to employ it in finiſhing ſome of his Works; which he might leave behind him, as a valuable Legacy both to his Family and the Public. He yielded ſo far to theſe Intreaties,

treaties, as to compoſe in a very ſhort time the following *Work, which he left perfect*, and tranſcribed fair for the Preſs.

I NEED not ſay much of the Nature of it, but refer the Reader to the Contents, and to the Book itſelf. I will only obſerve, it was chiefly intended for the Inſtruction of young Beginners, and for the uſe of thoſe that had ſuch under their Care. The Author has therefore been very explicit and accurate in the expoſition of every Part, and taken great care to expreſs his Sentiments in the moſt ſimple and eaſy manner. His Deſign was not only to complete a Courſe of Algebra, but alſo, as far as poſſible, to promote the Study of Geometry, by removing or explaining all thoſe Difficulties, which by his long Experience in teaching, he found were apt to retard, if not diſcourage young Students in their Progreſs through the Elements. And further, as Algebra is in it's own Nature an Art of Reaſoning, and may be conſidered as the Logical Inſtitutes of the Mathematician, the Author has been every where attentive to improve the Mind, and to furniſh it with every Method of Reaſoning that may be uſeful in our Reſearches into Nature. He has often expoſed the ſame Truths to us in ſeveral Lights, as we arrive at them by different Methods of Enquiry: ſince this ſerved to illuſtrate the Conſiſtency of thoſe Methods. He has alſo taken every Occaſion to obſerve the Tranſition of Truth from one Law to another: to obſerve the Conſiſtency of it's ſeveral Laws in the moſt intricate Caſes, where they ſeem moſt to thwart and contradict each other, as it Nature were put to her Shifts to preſerve that Conſiſtency and Uniformity which is every where the Characteriſtic of Truth. Such Obſervations cannot but ſuggeſt to the Mind the moſt ſimple and natural Ways of diſcovering Truth, and muſt therefore be greatly inſtructive, as well as entertaining, to all who are engaged in theſe Enquiries.

But the learned Mathematician muſt not wholly form his Judgment of the Author's Capacity from the following Work; which is of a Nature too low for the Exerciſe of ſo great a Genius. Yet the Reader will from hence learn to lament, with me, the Loſs of a Life ſo valuable to the learned World, when he was juſt entering upon Deſigns, which, had he lived to execute, would have greatly enriched our Treaſure of Mathematic Learning, and have proved Monuments to lateſt Poſterity worthy of his Name. His Manuſcripts were all left (for the Benefit of his Children) to the Care and Diſpoſal of John Robartes of *Twickenham* Eſquire, now Earl of Radnor; from whoſe Love and Eſteem for Letters and learned Men, and particularly the learned Author, the Public may be well aſſured, that theſe Remains will be ſo diſpoſed of, as to be moſt advantageous to Science, and honourable to their Author.

The Talents of Dr. Saunderson were not confined to the Study: when he put on the Companion, none ſupported Converſation with greater Wit and Elegance. His Diſcourſe was ſo enlivened with frequent Alluſions to Objects of Sight, that there appeared no Defect of the blind Man. Nothing was obſerved of that Diſreliſh of Humour, nothing of thoſe Abſences and Inattention to Diſcourſe, which uſually blemiſh and characteriſe Perſons devoted to theſe ſeverer Muſes. His Judgment on the various Paſſions and Intereſts of Mankind was equally acute as on the Subjects of Philoſophy. The Force and Spirit of his Expreſſion ſurpriſed and fixed the Attention of all that heard him. But above all, the Mathematician's Reverence for Truth ſhone forth in every Circumſtance of Life and Converſation, and added a Luſtre to his moſt ſhining Qualities. His Sentiments on Men and Opinions, his Praiſes or Cenſures, his Friendſhip or Diſregard were expreſſed without partiality or reſerve. This Frankneſs

of

of Temper endeared him to all ſuch as were happy in his Acquaintance and Eſteem; but raiſed him up what Enemies he had, and betrayed him to ſeveral Animoſities, which Men of more Art and Complaiſance would have choſe to avoid, at the Expence of ſo ſcrupulous and ſo diſintereſted a Sincerity.

IT would be thought an Omiſſion in theſe Memoirs of the Life of Dr. SAUNDERSON, if no notice were taken of the manner in which he reſigned it. The Reverend Mr. GERVAS HOLMES informed him, that the Mortification gained ſo much ground, that his beſt Friends could entertain no hopes of his Recovery. He received this notice of his approaching Death with great Calmneſs and Serenity; and after a ſhort Silence, reſumed Life and Spirits, and talked with as much Compoſure of Mind as he had ever done in his moſt ſedate Hours of perfect Health. He appointed the Evening of the following Day to receive the Sacrament with Mr. HOLMES; but before that came, he was ſeized with a Delirium, which continued to his Death.

Dr. *SAUNDERSON's*

PALPABLE ARITHMETIC

DECYPHER'D.

THE Author of the following Piece, (who is Dr. SAUNDERSON's *immediate Successor in the Professorship,) has been prevailed on to let it be inserted here, as an Illustration to some of his Performances; though it was originally designed for another Place.*

THAT the learned and ingenious Dr. SAUNDERSON, late *Lucasian* Professor of Mathematics in the University of CAMBRIDGE, notwithstanding the loss of his Sight, was able to make long and intricate Calculations, both Arithmetical and Algebraical, is a Thing as certain as it is wonderful. This appears beyond all Contradiction, not only from his elaborate Treatise of Algebra now published, but from other undoubted Monuments still in being. He had contrived for his own use, a commodious Notation for any large Numbers, which he could express on his *Abacus*, or Calculating Table, and with which he could readily perform any Arithmetical Operations, by the Sense of Feeling only; which therefore may be called his *Palpable Arithmetic.* As I have had an Opportunity, by the favour of Mrs. SAUNDERSON, of viewing and examining several Specimens of this Arithmetic, which by good fortune he had compleated and left behind him, though he has not left the least Hint by which his Method might be discovered; I had the Curiosity to propose to my-

self

ſelf the decyphering (as it may be called) of theſe Specimens, in which I have ſucceeded to my own Satisfaction. And as others may have the ſame Curioſity, or as this Method may poſſibly be of uſe to other Perſons, whoſe Misfortune may place them in like Circumſtances, I ſhall here attempt to give a ſuccinct but particular Account of it.

His Calculating Table was a ſmooth thin Board, ſomething more than a Foot ſquare, raiſed upon a ſmall Frame ſo as to lie hollow; which Board was divided by a great Number of equidiſtant parallel Lines, and by others as many, at right Angles to the former. The Edges of the Table were diſtinguiſhed by Notches, at about half an Inch diſtance from one another, and to each Notch belonged five of the aforeſaid Parallels; ſo that every ſquare Inch was divided into an Hundred little Squares. At every Point of Interſection the Board was perforated by ſmall Holes, capable of receiving a Pin; for it was by the help of Pins, ſtuck up to the Head through theſe Holes, that he expreſſed his Numbers. He uſed two Sorts of Pins, a larger and a ſmaller Sort; at leaſt their Heads were different, and might eaſily be diſtinguiſhed by feeling. Of theſe Pins he had a large Quantity in two Boxes, with their Points cut off, which always ſtood ready before him when he calculated. And theſe were his Inſtruments, of which we muſt now ſee the uſe.

In order to this we may firſt obſerve, that to every numeral Figure a little Square was appropriated on the Table, conſiſting of four of the little contiguous Squares above deſcribed, and which therefore allowed a ſmall Interval between each Figure; and this numeral Figure was different, according to the different Magnitude or Situation of the one or two Pins, which always compoſed it. For which Purpoſe he had ſettled in his Mind, and ſtrictly obſerved, the following Analogy or Notation. A great Pin in the Center of the Square (which,

(which, and no other, was always its Place,) was a Cypher, or o, and therefore I ſhall call it by that Name. Its chief Office was, to preſerve Order and Diſtance among his Figures and Lines. This Cypher was always preſent, except only in the Caſe of an Unit; to expreſs which the great Pin in the Center was changed into a little one. When 2 was to be expreſſed, the Cypher was reſtored to its Place, and the little Pin was put juſt over it. To expreſs 3, the Cypher remained as before, and the little Pin was advanced into the upper Angle on the right Hand. To expreſs 4, the little Pin deſcended, and immediately followed the Cypher. To expreſs 5, the little Pin deſcended to the lower Angle on the right Hand. To expreſs 6, the little Pin retreated, till it was juſt under the Cypher. To expreſs 7, the little Pin retreated into the lower Angle on the left Hand. To expreſs 8, the little Pin aſcended, till it was juſt before the Cypher. Laſtly, to expreſs 9, the little Pin aſcended into the upper Angle on the left Hand. And thus all the Digits were provided for, by an eaſy and uniform Notation, which might readily enough be apprehended and diſtinguiſhed by the Feeling. But to ſhew theſe Digits or Figures more diſtinctly, I ſhall endeavour to repreſent them by a Scheme. See *Fig.* I.

AND thus he could write down, as we may ſay, any propoſed Number upon his Table, and by lightly running his Fingers over it, he could at any time readily read it, and know what it ſignified. The great Pins or Cyphers, which were always placed at the Centers of the little Squares, and moſt frequently at equal Diſtances from one another, were a ſure Guide to direct him to keep the Line, to aſcertain the Limits of every Figure, and to prevent any Ambiguity that might otherwiſe ariſe. As three of the erect Parallels were ſufficient for a ſingle Figure, ſo three of the tranſverſe Parallels would ſuffice for a Line of Figures, and the next three

for

for another Line, and ſo on, without any Danger of interfering. And thus it is not hard to conceive, how he might have any Number of Lines of Figures upon his Table at the ſame time, in a deſcending order, or how he might derive one Number from another, or in a word, how he might make any Computations required. He could place and diſplace his Pins, as I have been informed, with incredible Nimbleneſs and Facility, much to the Pleaſure and Surprize of all the Beholders. He could even break off in the middle of a Calculation, and reſume it when he pleaſed, and could preſently know the Condition of it, by only drawing his Fingers gently over the Table. There is an obvious Expedient, which, eſpecially in long Calculations, would have made the Proceſs very expeditious, and therefore I queſtion not but he had often recourſe to it. And that is, to prepare the Table beforehand, (which any other might have done for him,) by filling every third Hole of every third parallel Line with large Pins, or Cyphers. Then when he intended to calculate, he would have nothing elſe to do, but to compleat every Figure, by adding a ſmall Pin in its proper Place. Except when an Unit was to be expreſſed, in which caſe he muſt have changed the large Pin into a ſmall one.

The Specimens of this Arithmetic which I have peruſed, and reduced to common Numbers, are certain Arithmetical Tables, which he had computed and preſerved for his own uſe; but for what Purpoſes they were calculated does not eaſily appear. They ſeem to have ſome relation to the Tables of natural Sines, Tangents, and Secants; but their full uſe I muſt leave to future Enquiry. They are four Pieces of ſolid Wood, of the Form of rectangular Parallelepipeds, each about eleven Inches long, five and an half broad, and ſomething above half an Inch thick. The two oppoſite Faces of every one were divided into little Squares, after the manner

of

of the *Abacus* above deſcribed, but they were perforated only in the neceſſary Places, where the Pins were ſtuck faſt up to the Head. Each Face exhibited nine ſmall Arithmetical Tables, of ten Numbers each; and every Number, generally ſpeaking, conſiſted of five Places or Figures. For farther ſatisfaction I have delineated one of theſe ſmall Tables, both as I found it, and as I have interpreted it. See *Fig.* II.

BUT beſides this Arithmetical Uſe of his Table, which was indeed its principal and primary Uſe, he could deſcribe upon it very neat and perfect Geometrical Figures, conſiſting of right Lines, any how interſecting one another, of which I have ſeen ſome Inſtances. This he did two ways, either by Pins ſet in Rows, which exhibited the appearance of pricked Lines; or by Pins placed only at the Interſections. Then by winding a Piece of fine Thread or Silk about their Heads, he could very well exhibit any continued ſtrait Lines at pleaſure, or any Syſtem of ſuch Lines. Whether he had palpable Letters alſo, ſomething like Printing Types, to diſtinguiſh the ſeveral angular Points, and to aſſiſt in demonſtrating the Properties of thoſe Figures, does not now appear. If he had found any occaſion for ſuch Helps, his fruitful Genius would eaſily have ſupplied them. It is not very difficult to conceive likewiſe, how the ſame Table might poſſibly be applied to the repreſenting all kinds of Algebraical Equations, and to the ſeveral Reductions of ſuch Equations, eſpecially by the uſe of the Types aforementioned, or ſomething like it. And he might have Types, in the nature of Pins, for the common Algebraic Signs, and to inſinuate the ſeveral Operations; by which means his Table would bear a near reſemblance to a Printer's Form: which I queſtion not but he could have read by his feeling only, if he had ever applied himſelf to it. I have been well aſſured, that he could ſpell very well; that he knew the Shapes of the Letters, both Small and

Capital,

Capital, and would ſometimes amuſe himſelf, when Opportunity offered, by reading the Inſcriptions upon Tomb-Stones with his Fingers. He has been often heard to regret very much, that he did not apply himſelf to learn to write in his younger Years, which (as he affirmed) he made no queſtion but he could have eaſily accompliſhed. None of theſe things will appear incredible to thoſe, who have known him give his Judgment with as much Accuracy upon the goodneſs of a Mathematical Inſtrument, and the juſtneſs of its Diviſions, by examining it only with his Touch, as the moſt judicious Eye could have done; and he was often conſulted on this Occaſion. This at leaſt is evident, that he could manage all kinds of Equations, and other perplext Calculations, with great Skill and Sufficiency; but how far he relied upon the Strength of his Imagination, (which was certainly very great,) and when he had recourſe to Mechanical Contrivances to aſſiſt it, I will not undertake to determine.

FROM what I have already deſcribed, and from many other ingenious Devices of the like Nature, which I have ſeen, I ſhall conclude with this general Obſervation, that as the knowledge and uſe of Symbols, (or of ſenſible and arbitrary Signs of intellectual Ideas,) is of the greateſt Importance and Extent in all Parts of the Mathematics; ſo he had invented a new Species of Mathematical Symbols, unknown and unheard of before, which were particularly accommodated to his own Circumſtances. The ſenſible Symbols, commonly received and made uſe of, to repreſent Mathematical Ideas, and to convey them to our own or to the Minds of others, are derived from two of our principal Senſes, and are either audible or viſible. The audible Symbols, it is true, he made good uſe of, and acquired much of his Knowledge by converſing with others, but chiefly by hearing others read all the beſt Mathematical Authors to him. But he was entirely

tirely deprived of the uſe of viſible Symbols, which to us we find are ſo abſolutely neceſſary, as that we always acquire the greateſt and moſt valuable Part of our Mathematical Knowledge by their Means. What did he do under this (as it ſhould ſeem) inſuperable Difficulty, in order to ſatisfy his great Thirſt after this kind of Knowledge? why, he had recourſe to another of his Senſes, which he had in great Perfection, and ſubſtituted Feeling in the place of Seeing; by inventing a Sort of Mathematical Symbols, which we may call palpable or tangible Symbols. Theſe he made uſe of, to convey thoſe Ideas to his Underſtanding, which were denied entrance through his Eyes. Thus by the aſſiſtance of theſe Symbols, imperfect and inadequate as they needs muſt be, and by the help of a quick Apprehenſion and obſtinate Perſeverance, he ſucceeded in theſe Sciences to Admiration. Theſe were the Inſtruments, as unfit as they may ſeem, by which he conveyed to his Mind the moſt abſtruſe and ſublime Mathematical Ideas, and by which he was enabled to deduce from them the moſt general and uſeful Concluſions. This we muſt all confeſs, if we have any degree of Candour, as well as Eyes and Underſtanding, to be very extraordinary and ſurprizing. I cannot but repreſent to my Imagination, during the whole courſe of his Studies, the Idea of a great Mathematical Genius, ſtruggling under the greateſt Diſadvantages, and labouring under the ſevereſt natural Inabilities and Diſcouragements. Yet whatever Difficulties he meets in his way, by a happy Sagacity and ſtubborn Induſtry, he finds as many Expedients to overcome them all; reſolving not to relinquiſh his purſuit, but ſteadily to perſiſt, till he is ſuperior to all Obſtacles, and till he has gratified the no mean Ambition, of placing himſelf in the firſt Claſs of Mathematicians.

A Second Advertisement.

SINCE the printing of the former Advertisement, I have been favoured with the use of an English *Translation of* Napeir's *Book, entitled* A Description of the admirable Table of Logarithmes, *perused and approved by the Author himself, and printed in the year* 1616. *From which Book it appears, that He himself, and not* Briggs, *was likewise the inventer of the System of Logarithms now in use, as I had before observed. For thus he writes,* book 1. chap. 4. sect. 9. pag. 19, 20. "*But because the addition and subtrac-*" *tion of these former numbers may seeme somewhat painfull, I intend (if it*" *shall please God) in a second Edition, to set out such* Logarithmes *as shal*" *make those numbers aboue written to fall upon decimal numbers, such as*" 100 000 000, 200 000 000, 300 000 000, *&c. which are easie to be ad-*" *ded or abated to or from any other number.*" *Another quotation to the same purpose, from the Author's* Rabdologia, *published in the year* 1617, *may be seen in Mr. Professor* WARD's Lives of the Professors of Gresham College, p. 122.

Amongst Professor SAUNDERSON's Questions for Exercise in the Rule of Three, *there are some which belong to the Double Rule of Three, where five numbers are given in order to find a sixth; as in the thirtysecond question, pag.* 13. If two men in three days will earn four shillings, how much will five men earn in six days? *Where the numbers given are* 2, 3, 4, 5, 6; *the three first of which are always in the conditional part of the question, and the other two in the remaining part, which moves the question. It is also to be noted, that in all questions belonging to the Double Rule of Three, the numbers are so to be placed, as that the first and the fourth, the second and the fifth, the third and the number sought, may be of the same denomination respectively; as in the question proposed:* 2 men, 3 days, 4 shillings; 5 men, 6 days, shillings required.

These things being premised, the Professor's first rule is this, pag. 14. In all questions of this nature, if the three last numbers be multiplied together, and the product be divided by the product of the two first, the quotient will give the number sought. *As in the question proposed,* $\frac{4\times5\times6}{2\times3}=20$, *the number sought. Again, in the thirtythird question,* If for the carriage of 3 hundred weight 40 miles I must pay 7 shillings and 6 pence, what must I pay for the carriage of 5 hundred weight 60 miles? *The terms are* 3 hundred weight, 40 miles, $7\frac{1}{2}$ shillings; 5 hundred weight, 60 miles. *Answ.* $\frac{7\frac{1}{2}\times5\times60}{3\times40}=18\frac{3}{4}$, *that is,* 18 *shillings and* 9 *pence.*

There is another case, wherein inverse proportion is partly concerned, as in the thirtyninth question, pag. 16. If two acres of land will maintain three horses four days, how long will five acres maintain six horses? *Here it is plain, that inverse proportion is concerned: for the same quantity of land will maintain six horses but half the time that it would maintain three. Wherefore for resolving questions of this nature, the Professor proposes a second rule, pag.* 17, 18. In all questions belonging to the Double Rule of Three inverse, where the numbers are supposed to be ordered as in the Double Rule of Three direct, if the three middle numbers be multiplied together, and the product be divided by the product of the two extremes, the quotient of this division will be the number sought. *As in the thirtyninth question, the order of the terms being* 2 acres, 3 horses, 4 days; 5 acres, 6 horses; *therefore* $\frac{3 \times 4 \times 5}{2 \times 6} = 5$, *the number of the days sought.*

But here it must be observed, that the terms of this question are capable of being ranged in a different order, thus: If three horses will eat up the grass of two acres of land in four days, how long will six horses be in eating up that of five acres? *Here the order of the terms given is,* 3 horses, 2 acres, 4 days; 6 horses, 5 acres; *according to which order, neither this rule nor the former will solve the question.*

Wherefore, to clear this difficulty, and to find out, in what order the Professor's second rule requires the terms given should be placed, it may be useful to investigate a general Theorem for resolving all questions in the Double Rule of Three. This I shall do in the method proposed by the late Mr. Ward *of* Chester, *in his* Introduction to the Mathematicks, part 1. chap. 7. sect. 3. pag. 96, *where he denotes the six quantities in questions belonging to the Double Rule of Three, by these six letters,* P, T, G, p, t, g; *the capitals* P, T, G *denoting the terms in the conditional part of the question, as the small letters* p, t, g *do the other terms, of the same denomination with their respective capitals.* P, p *signify the principal terms, or causes of the gain, loss, or expence mentioned in the question;* T, t *the times (as in quest.* 32 *and* 39*) or spaces (as in quest.* 33*) wherein the said gain, loss, or expence is produced; and* G, g *the gain, loss, or expence itself, arising from the principals in the aforesaid times or spaces.*

Now in order to form a Theorem for resolving questions in the Double Rule of Three direct, let us take any particular question, as for example, the thirty-second: If 2 men in 3 days will earn 4 shillings, how much will 5 men earn in 6 days? *Where* 2=P, 3=T, 4=G, 5=p, 6=t, *and* g *is the number sought. This question then may be resolved into two proportionalities, thus:* If 2 (*P*) men will earn 4 (*G*) shillings in a given time, 5 (*p*) men will earn 10 $\left(\frac{Gp}{P}\right)$ shillings in the same time. *Again,* If in 3 (*T*) days

10 $\left(\frac{Gp}{P}\right)$ shillings be earned, then in 6 (t) days 20 $\left(\frac{Gpt}{PT}\right)$ will be earned. *Therefore* $\frac{Gpt}{PT} = g$; *and multiplying both sides by* PT, *we have the Theorem* $Gpt = gPT$. *And since the thirtysecond question proposed, gives us five terms in this order*, P, T, G, p, t, *the term sought being* $g = \frac{Gpt}{PT}$, *it is manifest, that the term sought is found, by dividing the product of the three last terms given, by the product of the two first, according to the Professor's first rule.*

Let us now take the thirtyninth question, that we may thence form a Theorem for resolving questions of the same nature with it, which belong to the Double Rule of Three inverse. If 2 (G) acres of land will maintain 3 (P) horses 4 (T) days, how long will 5 (g) acres maintain 6 (p) horses? *Which question I resolve into two proportionalities: first into this inverse proportionality*, If a certain piece of ground will maintain 3 (P) horses 4 (T) days, it will maintain 6 (p) double the number of horses but half the time, that is, 2 $\left(\frac{PT}{p}\right)$ days: *Secondly, into this direct one*, If 2 (G) acres will maintain this double number of horses 2 $\left(\frac{PT}{p}\right)$ days, 5 (g) acres will maintain them 5 $\left(\frac{gPT}{Gp}\right)$ days. *Therefore* t, *the number sought, is* $\frac{gPT}{Gp}$. *And multiplying both sides by* Gp, *we have the same Theorem as before*, $Gpt = gPT$, *which therefore is a general Theorem for solving all questions in the Double Rule of Three, whether direct or inverse. And since the order of the terms given in the thirtyninth question is* G, P, T, g, p; *and since also* t, *the number sought, is found equal to* $\frac{gPT}{Gp}$, *therefore according to the Professor's second rule, if the product of the three middle terms be divided by the product of the two extremes, the quotient will be the term sought.*

But in the practising of this second rule, care must be always taken to make the term G *the first in order: and then, if* t *be sought, as in the thirtyninth question, the order of the terms given will be* G, P, T, g, p, *as before; but if* p *be sought, the order will be* G, T, P, g, t. *And lastly, I think I scarce need to observe, that if* g *be the term sought, the operation must always be made by the Professor's first rule, where the order of the terms is* P, T, G, p, t, *as in the thirtysecond and thirtythird questions; or else* T, P, G, t, p, *which comes to the same thing.*

The

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ERRATA.

In the laſt Page of the former Advertiſement, lin. 34, 35. *read* compleret.
* Pag. 120. Prob. 23. lin. 2. for 16 *years* read 15 *years*.

POSTU-

Fig: I.

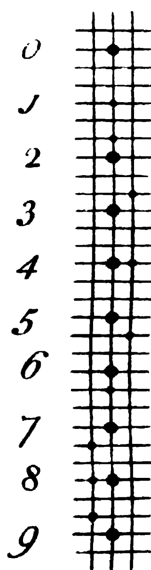
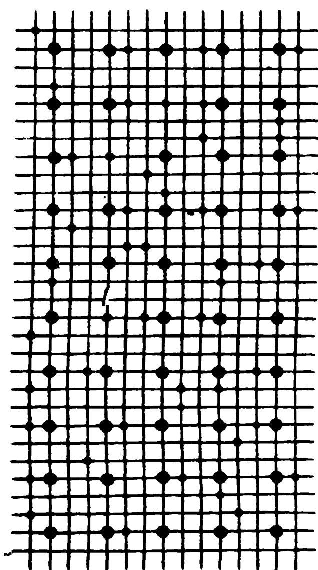


Fig: II.



94084
24186
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63968
71880
78568
84358
89464
94030

Palpable Arithmetic fronting Page xxiv.

POSTULATA.

BEFORE I enter upon my province, it may not be amiſs to acquaint my young diſciple what preparations he is to make, and what qualifications I expect of him beforehand, that we may neither of us find ourſelves diſappointed afterwards. I expect then that he knows how to add, to ſubtract, to multiply, to divide, to find a fourth proportional, and to extract roots, eſpecially the ſquare root: nay I expect further, that he ſhall not only be able to perform all theſe operations exactly and readily, but alſo that he ſhall be able to apply them upon all common occaſions; in a word, I expect that he be tolerably well ſkilled in common Arithmetick, at leaſt ſo far as relates to whole numbers: for this reaſon it is that I have prefixed a few arithmetical queſtions, wherein he may firſt try his ſtrength and ſkill before he ventures any further; they are for the moſt part very eaſy, I cannot ſay indeed they are the beſt choſen, but they were ſuch as lay in my way when I firſt begun this work and was haſtening to matters of greater moment, and I do not ſee but they may, if ſtudied with care and attention, anſwer well enough the end they were intended for: If he finds no difficulty in theſe, he will have little reaſon to doubt of his ſucceſs afterwards; but if he does, he ought then at laſt to become ſenſible of his own defects and to endeavour to ſupply whatever is wanting, and to correct whatever is amiſs before he enters himſelf under my conduct; in the mean time he has my leave to hope that I ſhall be leſs upon the reſerve with him when he falls more immediately under my care.

N. B. The *praxis* of the rule of proportion, and of the rule for extracting the ſquare root, not being (properly ſpeaking) of the nature of ſimple *poſtulata*, but rather deducible from the four firſt; I ſhall not fail to demonſtrate theſe rules ſo ſoon as I ſhall find proper opportunities for that purpoſe.

Questions for exercise in Multiplication.

Multiplication is taking any one number called the multiplicand as often as is expressed by any other number called the multiplicator, and the number produced by this operation is called the product: whence it follows, that the product contains the multiplicand as often as there are units in the multiplicator, and that if a number of a greater denomination is to be reduced to an equivalent number of a less, it must be done by multiplication. As for example; In a pound sterling there are 20 shillings; therefore in every sum of money consisting of even pounds, there are twenty times as many shillings as there are pounds; therefore if any number of pounds be multiplied by 20, the product will be an equivalent number of shillings; and the same must be observed in all other cases.

QUEST. 1.

It is required to reduce 456 pounds, 13 shillings and 4 pence into shillings, pence and farthings.

Answer.	Shillings	9133
	Pence	109600
	Farthings	438400.

QUEST. 2.

A certain island contains 36 counties, every county 37 parishes, every parish 38 families, and every family 39 persons: I demand the number of parishes, families and persons in the whole island.

Answer.	Parishes	1332
	Families	50616
	Persons	1974024.

QUEST. 3.

In 1730 years, 42 weeks and 3 days, how many minutes?

N. B. A year consists of 365 days 6 hours, and an hour of 60 minutes.

Hours in one year	8766
In 1730 years	15165180
In 42 weeks 3 days	7128
In the whole	15172308
Minutes in the whole	910338480.

QUEST. 4.

There is a certain field 102004 *feet long, and* 102003 *feet broad: I demand the number of ſquare feet therein contained.*

Anſwer. 10404714012.

QUEST. 5.

There is a certain floor 24 *feet,* 4 *inches broad, and* 96 *feet,* 6 *inches long: I demand how many ſquare inches are therein contained.*

Anſwer. 338136 ſquare inches.

QUEST. 6.

A certain piece of wood 1 *foot,* 2 *inches thick,* 3 *feet,* 4 *inches broad, and* 5 *feet,* 6 *inches long, is to be cut into ſmall cubes like dies, each of which is to be a quarter of an inch every way: I demand into how many dies the whole may be reſolved.*

Anſwer. The whole may be reſolved into 2365440 dies.

QUEST. 7.

I demand the number of changes that may be rung on 12 *bells.*

Changes upon	2 bells	2
on	3 bells	6
on	4 bells	24
on	5 bells	120
on	6 bells	720
on	7 bells	5040
on	8 bells	40320
on	9 bells	362880
on	10 bells	3628800
on	11 bells	39916800
on	12 bells	479001600.

QUEST. 8.

How many different ways can four common dies come up at one throw?

Answer 1296 ways.

QUEST. 9.

Suppose one undertakes to throw an ace at one throw with four common dies; what probability is there of his effecting it?

Answer. By the last question four dies can come up 1296 different ways with and without the ace; and by a like computation, they can come up 625 ways without the ace; therefore there are 671 ways wherein one or more of them may turn up an ace; therefore the undertaker has the better of the lay in the proportion of 671 to 625.

QUEST. 10.

There are two inclosures of the same circumference, that is, both inclosed with the same number of pales; but one is a square whose side is 125 feet, and the other an oblong or long square, 124 feet in breadth, and 126 in length: quære *which is the greater close, that is, which,* cæteris paribus, *will bear most grass.*

Answer. The square: for that contains 15625 feet; whereas the other contains but 15624.

Questions for exercise in Division.

The design of division is to shew how often one number called the divisor is contained in another called the dividend, and the number that shews this is called the quotient: whence, and from the definition of multiplication already given, I observe 1*st*, That the divisor multiplied by the quotient, and consequently the quotient multiplied by the divisor, will always be equal to the dividend, provided there be no remainder after the division is over; but if there be, then this remainder added to, or taken into the product will give the dividend, which is the best proof of division. 2*dly*, That as the divisor is such a part of the dividend as is expressed by the quotient; so also is the quotient such a part as is expressed by the divisor. Thus 12 divided by 3 quotes 4; therefore 3 is a fourth part, and

4 a third part of 12. *3dly*, Hence may a number be found that ſhall be diviſible by any two given numbers whatever without remainders, to wit, by multiplying the two given numbers together. Thus if I would have a number that can be divided by both 6 and 9 without any remainders, I multiply 9 by 6, and the product 54 will anſwer both conditions; though 18 be the leaſt number of that kind. *4thly*, Multiplication and diviſion by the ſame number are the reverſe of each other, and ſo muſt neceſſarily have contrary effects: for whereas multiplication increaſes a number, by taking it as often as is expreſſed by the multiplicator, diviſion (on the contrary) leſſens it, by taking only ſuch a part of it as is expreſſed by the diviſor. *5thly*, Hence if a number of a leſſer denomination be to be changed into an equivalent number of a greater, as farthings into pence, pence into ſhillings &c, it muſt be done by diviſion, as the reverſe is done by multiplication. *6thly*, Whenever it is propoſed to know how often one quantity of any kind is contained in another of the ſame kind, the numbers repreſenting theſe quantities muſt be reduced to the ſame denomination before any diviſion can take place. Thus if I would know how many thirteenpencehalfpennies there are in 20 ſhillings, I muſt not only reduce the thirteenpencehalfpenny to 27 halfpence, but alſo the 20 ſhillings into 480 halfpence; and then muſt enquire by diviſion how often 27 halfpence are contained in 480 halfpence, that is, how often 27 is contained in 480; the quotient is 17, and the remainder 21, that is, 21 halfpence; for in all diviſion, the remainder muſt be of the ſame denomination with the dividend whereof it is a part; therefore in 20 ſhillings there are 17 thirteenpencehalfpennies, and 10 pence halfpenny over.

QUEST. 11.

It is required to reduce 987654321 *farthings into pounds, ſhillings and pence.*

Anſwer. 987654321 farthings are equivalent to 246913580 pence and 1 farthing; or to 20576131 ſhillings, 8*d.* 1*q*; or to 1028806 pounds, 11*s.* 8*d.* 1*q.*

QUEST. 12.

One lends me 1296 *guineas when they were valued at* 1*l.* 1*s. and ſixpence apiece: how many muſt I pay him when they are valued at* 1. 1*s. apiece?*

Anſwer. 1326 guineas, 18 ſhillings.

QUEST.

QUEST. 13.

A certain floor 24 feet 4 inches broad, 96 feet 6 inches long, is to be laid at the rate of 12 pence the square foot: I demand what the whole charge will amount to.

Answer. The floor contains 338136 square inches, or 2348 square feet and 24 square inches; therefore the whole charge amounts to 117 pounds, 8 shillings and two pence.

QUEST. 14.

There is a certain cooler 36 inches deep, 42 inches wide, and 72 inches long: I demand its solid content in English gallons.

Note. An ale gallon is 282 cubic inches.

Answer. The vessel contains 108864 cubic inches, that is, 386 gallons and 12 cubic inches over.

QUEST. 15.

A cubic foot of water weighs 76 pounds, Troy or Roman weight; and air is 860 times lighter than water: I demand the weight of a cubic foot of air.

N. B. A pound Troy contains 12 ounces, one ounce 20 pennyweights, and one pennyweight 24 grains.

Answer. A cubic foot of air weighs Troy weight 1 *oz.* 1 *pwt.* 5 *gr.*

QUEST. 16.

The mean time of a lunation, that is, from new moon to new moon, is 29 days, 12 hours, 44 minutes and 3 seconds; and a Julian year consists of 365 days, 6 hours: I demand then how many lunations are contained in 19 Julian years.

Hours in a Lunation	708
Minutes	42524
Seconds	2551443
Hours in 19 Julian years	166554
Minutes	9993240
Seconds	599594400

Lunations 235; and 1 hour, 28′, 15″ over.

QUEST. 17.

In what time may all the changes on 12 bells be rung, allowing 3 seconds to every round? See Quest. the *7th*.

The number of changes on 12 bells 479001600
The time 1437004800 seconds,
or 23950080 minutes,
or 399168 hours,
or 45 years, 27 weeks, 6 days, 18 hours.

QUEST. 18.

A General of an army distributes 15 pounds, 19 shillings and 2 pence half-penny, among 4 captains, 5 lieutenants and 60 common soldiers, in the manner following: Every captain is to have 3 times as much as a lieutenant, and every lieutenant twice as much as a common soldier: I demand their several shares.

The share of a common soldier	3*s*. 4*d*. $\frac{3}{4}$
of a lieutenant	6*s*. 9*d*. $\frac{1}{2}$
of a captain	1*l*. 0*s*. 4*d*. $\frac{1}{2}$

Questions for exercise in the Rule of Three.

And first in the Rule of Three Direct.

The rule of proportion, or rule of three, or by some the golden rule, is that which teacheth, having three numbers given to find a fourth proportional, that is, to find a fourth number that shall have the same proportion to some one of the numbers given, as is expressed by the other two; and therefore whenever a question is proposed wherein such a fourth proportional is required, that question is said to belong to the rule of proportion. Now in questions of this nature, especially where the numbers given are not merely abstract numbers, but are applied to particular quantities, three things are usually required, to wit, preparation, disposition, and operation.

First as to the preparation, it must be observed that of the three numbers given in the question, two will always be of the same kind, and must be reduced to the same denomination, if they be not so already; and

and if the remaining number be of a mixt denomination, that alſo muſt be reduced to ſome ſimple one.

Secondly, in diſpoſing the numbers thus prepared, thoſe two that are of the ſame denomination muſt be made the firſt and third numbers in the rule of proportion, and conſequently the remaining number muſt be the ſecond. But here particular care muſt be taken, that of the two numbers that are of the ſame denomination, that be made the third in the rule of proportion, upon which the main ſtreſs of the queſtion lies, or to which the queſtion more immediately relates, or which contains the demand; and the place of this number being once known, the other two muſt take their places as above directed. This ordering of the numbers for the operation is commonly called, ſtating of the queſtion.

Laſtly, having thus ſtated the queſtion, multiply the ſecond and third numbers together; divide the product by the firſt, and the quotient thence ariſing will be the fourth number ſought; which fourth number, as well as the remainder, if there be any, muſt always be underſtood to be of the ſame denomination with the ſecond. As for example,

QUEST. 19.

A piece of plate weighing 3 pounds, 4 ounces and 5 pennyweights, Troy weight, is valued at 5 ſhillings and 6 pence an ounce; what is the value of the whole?

Here we have three quantities concerned in the queſtion, *viz.* 3 pounds, 4 ounces and 5 pennyweights; one ounce; and 5 ſhillings and 6 pence; whereof the two firſt, which are of the ſame kind, muſt be reduced to the ſame denomination, and the laſt to a ſimple one, thus: for one ounce I write 20 pennyweights; for 3 pounds, 4 ounces and 5 pennyweights, 805 pennyweights; and for 5 ſhillings and 6 pence, 66 pence; and ſo the numbers are ſufficiently prepared. In the next place I enquire which of the two numbers 20 and 805, which are of the ſame denomination, is that upon which the main ſtreſs of the queſtion lies, and I find it to be 805; for the main buſineſs of this queſtion is to enquire into the value of 805 pennyweights of plate; the reſt being no more than *data* in order to diſcover this: So I make 805 my third number, 20 which is a number of the ſame denomination my firſt, and 66 my ſecond, and ſtate the queſtion thus; *If 20 pennyweights of plate be worth 66 pence, what will 805 pennyweights of plate be worth?* Now to anſwer this queſtion, I multiply 805 by 66, and the product is 53130; this I divide by my firſt number 20, and the quotient is 2656, and there remains 10, that is, 10 pence; therefore to render my quotient more compleat, I bring the remaining 10 pence

pence into 40 farthings, and ſo divide again by 20, and find the quotient to be 2, that is, 2 farthings, without any remainder; ſo the value ſought is 2656 pence, 2 farthings; that is, 11 pounds, 1 ſhilling and 4 pence halfpenny.

A demonſtration of this Praxis.

Caſe 1ſt. Now to demonſtrate this manner of operation, I ſhall reſume the foregoing queſtion, but at firſt under a different ſuppoſition, as thus; *If one pennyweight of plate coſt* 66 *pence, what will* 805 *pennyweights coſt?* Here nobody doubts but that upon this ſuppoſition, 805 pennyweights will coſt 805 times 66 pence, or 66 times 805, that is, 53130 pence; therefore in all inſtances of this kind, that is, where the firſt number in the rule of proportion is unity, the fourth number muſt be found by multiplying the ſecond and third numbers together.

Caſe 2d. Let us now put the queſtion as it was at firſt ſtated, to wit, *If* 20 *pennyweights of plate be worth* 66 *pence, what will* 805 *pennyweights be worth?* Now upon this ſuppoſition it is eaſy to ſee, that neither 1 pennyweight, nor conſequently 805 pennyweights will be worth above a 20th part of what they were in the former caſe; and therefore we muſt not now ſay that 805 pennyweights are worth 53130 pence, but a 20th part of that ſum, *viz.* 2656 pence 2 farthings: and as this way of reaſoning will be the ſame in all other inſtances, it follows now, that *In the rule of proportion, let the numbers given be what they will, the fourth number muſt be had by multiplying the ſecond and third numbers together, and dividing the product by the firſt.* Q. E. D.

QUEST. 20.

How far will one be able to travel in 7 *days* 8 *hours, at the rate of* 13 *miles every* 4 *hours, allowing* 12 *hours to a travelling day?*

Anſwer. 299 miles.

QUEST. 21.

What will 1296 *yards of walling amount to, at the rate of* 4 *ſhillings and* 5 *pence a rod, a rod being* 5 *yards and a half?*

Anſwer. 52 pounds, 8 pence, 3 farthings.

QUEST. 22.

In the mint of England *a pound of gold, that is,* 11 *ounces fine and* 1 *allay, is at this time coined into* 44 *guineas and an half: I demand how much ſterling a pound of pure gold is worth, obſerving that the allay is valued at nothing.*

Anſwer. 50 pounds, 19 ſhillings and 5 pence $\frac{1}{2}$ penny

QUEST. 23.

What is the annual intereſt of 987 *pounds,* 6 *ſhillings and* 5 *pence at the rate of* 6 per cent?

Anſwer. 59 pounds, 4 ſhillings and 9 pence $\frac{1}{2}$ penny.

QUEST. 24.

The circumference of the earth according to the French menſuration is 123249600 *French feet: I demand the ſame in Engliſh miles.*

N. B. A thouſand French feet are equivalent to 1068 Engliſh feet; 3 feet make a yard, and 1760 yards make a mile.

Anſwer.	131630573	Engliſh feet,
or	43876857	yards and 2 feet,
or	24930	miles, 57 yards and 2 feet.

QUEST. 25.

Suppoſing all things as in the foregoing queſtion, I demand how long a ſound will be in paſſing from pole to pole, upon a ſuppoſition that a ſound paſſes over 1142 *feet in a ſecond of time.*

Anſwer. 16 hours and 32 ſeconds.

QUEST. 26.

Monſieur Huygens *found that at* Paris, *the length of a pendulum that ſwung ſeconds, was three feet,* 8 *lines and* $\frac{1}{2}$: *I demand it's length in Engliſh meaſure.*

Note.

Note. A line is $\frac{1}{12}$ part of an inch, and 1000 French half lines are equivalent to 1068 English ½ lines, as in the 24th queſtion.

Anſwer. The length in Engliſh meaſure of a pendulum that ſwings ſeconds, is 941 Engliſh ½ lines; or 39 inches, 2 lines and ½.

QUEST. 27.

I demand in how long a time, a pipe that diſcharges 15 *pints in* 2 *minutes,* 34 *ſeconds, will fill a ciſtern that is* 36 *inches deep,* 42 *inches wide, and* 72 *inches long.* (ſee queſtion the 14th.)

Anſwer. In 31707 ſeconds; or 8 hours, 48′. 27″.

For as eight pints make a gallon, ſo alſo eight cubic half inches, that is, eight ſmall cubes of half an inch every way make one cubic inch; therefore a pint contains 282 cubic half inches, and fifteen pints 4230; but the whole veſſel contains 108864 cubic inches by queſt. 14; which are equivalent to 870912 cubic half inches; therefore this queſtion ought to be ſtated thus;

If 4230 *cubic half inches be diſcharged in* 154 *ſeconds of time, in what time will* 870912 *cubic half inches be diſcharged?* And the anſwer is,

In 8 hours, 48′. 27″. as above.

QUEST. 28.

If a wall 6 *feet thick,* 9 *feet high and* 432 *feet long, coſt* 720 *pounds in building, what will be the price of a wall of the ſame materials, that is* 12 *feet thick,* 18 *feet high and* 576 *feet long?*

In the former wall are contained 23328 cubic feet; in the latter 124416; therefore the anſwer to this queſtion is 3840 pounds.

QUEST. 29.

A certain ſteeple projected upon level ground a ſhadow to the diſtance of 57 *yards, when a four-foot ſtaff perpendicularly erected caſt a ſhadow of* 5 *feet* 6 *inches: what was the height of the ſteeple?*

Anſwer. 41 yards, 1 foot, 4 inches.

QUEST. 30.

Two perſons A *and* B *make a joint ſtock*; A *puts in* 372 *pounds*, *and* B 496 *pounds*, *for the ſame time*; *and they gain* 114 *pounds*, 2 *ſhillings*: *I demand each mans ſhare of the gain.*

Both their ſtocks make 868 pounds: ſay then, if 868 pounds ſtock, bring in 114 pounds, 2 ſhillings gain, what will 372 pounds, *A*'s part of the ſtock bring in? *Anſwer.* 48 pounds, 18 ſhillings for *A*'s ſhare of the gain; and this ſubtracted from the whole gain, leaves 65 pounds, 4 ſhillings, for *B*'s ſhare of the gain.

Note. If there be ever ſo many partners, their ſhares of the gain muſt all be found by the rule of proportion, except the laſt, which may be had by ſubtraction; but it would be better to find them all by the rule of proportion, becauſe then, if all the ſhares when added together, make up the whole gain, it will be an argument that the work is rightly performed.

QUEST. 31.

Two perſons A *and* B *make a joint ſtock*; A *puts in* 496 *pounds for* 2 *months*, *and* B 620 *pounds for* 3 *months*; *and they gain* 456 *pounds. What will be each mans ſhare of the gain?*

In order to give an anſwer to this queſtion, it muſt be conſidered, that it is the ſame in the caſe of trade, as it is in that of money let out to intereſt, where time is as good as money, that is, whoever lets out 496 pounds for 2 months, is intitled to the ſame ſhare of the whole gain, as if he had let out twice as much, that is, 992 pounds, for one month: in like manner, he that lets out 620 pounds for 3 months, has a right to the ſame ſhare of the gain, as if he had let out three times as much, that is, 1860 pounds, for 1 month: ſubſtitute therefore theſe ſuppoſitions inſtead of thoſe in the queſtion, which may ſafely be done without affecting the concluſion, and then this queſtion will be reduced to the form of the laſt, without any conſideration of the particular quantity of time, thus; *Two merchants* A *and* B *make a joint ſtock*; A *puts in* 992 *pounds, and* B 1860 *pounds for the ſame time*; *and they gain* 456 *pounds. What will be their reſpective ſhares of the gain?*

Anſwer. *A*'s ſhare will be 158 pounds, 12 ſhillings and 2 pence; and *B*'s, 297 pounds, 7 ſhillings and 10 pence.

QUEST.

QUEST. 32.

If two men in three days will earn 4 ſhillings, how much will 5 men earn in 6 days?

This and the following queſtion belong to that which they call the double rule of three, wherein 5 numbers are concerned: theſe numbers muſt always be placed as they are in this example, that is, the two laſt numbers muſt always be of the ſame denomination with the two firſt reſpectively, and the number ſought of the ſame denomination with the middle one; then may the queſtion be reduced to the ſingle rule of three two ways, either by expunging the firſt and fourth numbers, or the ſecond and fifth: if you would have the firſt and fourth numbers expunged, you muſt argue thus; two men will earn as much in three days, as one man in two times 3, or 6 days; alſo 5 men will earn as much in 6 days, as one man in 30 days; ſubſtitute therefore this ſuppoſition and this demand, inſtead of thoſe in the queſtion, and it will ſtand thus; If one man in 6 days will earn four ſhillings, how much will one man earn in 30 days? Which is as much as to ſay, *If in 6 days a man will earn 4 ſhillings, how much will he earn in 30 days?*

Anſwer. 20 ſhillings.

If you would have the ſecond and fifth numbers expunged, you muſt argue thus; two men will earn as much in three days, as 3 times two or 6 men in one day; alſo 5 men will earn as much in 6 days, as 30 men in one day; put then the queſtion this way, and it will ſtand thus; If 6 men in one day will earn 4 ſhillings, how much will 30 men earn in one day? That is, *If in any quantity of time 6 men will earn 4 ſhillings, how much will 30 men earn in the ſame time?*

Anſwer. 20 ſhillings, as before.

Whoſoever attends to both theſe methods of extermination, will eaſily fall into a third, which includes both the other, and in practice is much better than either of them; for at the concluſion of both operations, the number ſought was found by multiplying 30 by 4, and dividing the product by 6: Now if he looks back, and traces out theſe numbers, he will find that the number 30 came from the multiplication of the two laſt numbers 5 and 6 together, that 4 was the middle number in the queſtion, and that the diviſor 6 was the product of the two firſt numbers

2 and

2 and 3 multiplied together; therefore *In all queſtions of this nature, if the three laſt numbers be multiplied together, and the product be divided by the product of the two firſt, the quotient will give the number ſought, without any further trouble.*

QUEST. 33.

If for the carriage of three hundred weight 40 *miles, I muſt pay* 7 *ſhillings and* 6 *pence, what muſt I pay for the carriage of* 5 *hundred weight* 60 *miles?*

Anſwer. 225 pence, or 18 ſhillings and 9 pence.

Queſtions in the rule of three Inverſe.

Hitherto we have inſtanced in the rule of three direct; but there is alſo another rule of proportion, called the rule of three inverſe; which as to the preparation, and diſpoſition of it's numbers, differs nothing from the rule of three direct, but only in the operation; for whereas there, the fourth number was found, by multiplying the ſecond and third numbers together, and dividing by the firſt; here it is found by multiplying the firſt and ſecond numbers together, and dividing by the third. All that remains then, is to be able to diſtinguiſh, when a queſtion belongs to one rule, and when to the other; in order to which, obſerve the following directions: *If more requires more, or leſs requires leſs, work by the rule of three direct; but if more requires leſs, or leſs requires more, work by the rule of three inverſe.* The meaning whereof is, that if, when the third number is greater than the firſt, the fourth muſt be proportionably greater than the ſecond; or if, when the third number is leſs than the firſt, the fourth muſt be proportionably leſs than the ſecond, the queſtion then belongs to the rule of three direct: But if, when the third number is greater than the firſt, the fourth muſt be leſs than the ſecond; or when the third number is leſs than the firſt, the fourth muſt be greater than the ſecond; in either of theſe caſes, the queſtion belongs to the rule of three inverſe, and muſt be reſolved as above directed.

As for example,

QUEST. 34.

If 12 *men will eat up a quantity of proviſion in* 15 *days, how long will* 20 *men be in eating up the ſame?*

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This queſtion is of ſuch a nature, that more requires leſs; for 20 men will conſume the ſame proviſion in leſs time than 12; therefore the queſtion belongs to the rule of three inverſe; ſo I multiply the firſt and ſecond numbers together, and divide by the third, and the quotient 9, that is, 9 days, is an anſwer to the queſtion.

A demonſtration of the rule of three Inverſe.

If I was to anſwer this queſtion by pure dint of thought, without any rule to direct me, I ſhould reaſon thus: whatever quantity of proviſion laſts 12 men 15 days, the ſame will laſt 1 man 12 times as long, that is, 12 times 15, or 180 days; but if it will laſt 1 man 180 days, it will laſt 20 men but the 20th part of that time, that is, 9 days: here then the fourth number was found by multiplying the firſt and ſecond numbers together, and dividing the product by the third; and the reaſon is the ſame in all other caſes, wherever the rule of three inverſe is concerned. *Q. E. D.*

QUEST. 35.

One lends me 372 *pounds for* 7 *years and* 8 *months, or* 92 *months: how long muſt I lend him* 496 *pounds for an equivalent?*

Anſwer. 5 years, 9 months.

QUEST. 36.

If a ſquare pipe 4 *inches and* 5 *lines wide, will diſcharge a certain quantity of water in one hour's time; in what time will another ſquare pipe,* 1 *inch and* 2 *lines wide, diſcharge the ſame quantity from the ſame current?*

The orifice of a ſquare pipe 4 inches, 5 lines, or 53 lines wide, contains 2809 ſquare lines; and the orifice of a pipe 1 inch, 2 lines, or 14 lines wide, contains 196 ſquare lines. Say then, *If an orifice of* 2809 *ſquare lines will diſcharge a certain quantity of water in one hour; in what time will an orifice of* 196 *ſquare lines diſcharge the ſame?*

Anſwer. In 14 hours, 19′. 54″.

QUEST.

QUEST. 37.

If 3 men, or 4 women, will do a piece of work in 56 days, how long will one man and one woman be in doing the same?

Becauſe of the 3 men, or 4 women, ſome number muſt be found that is divisible both by 3 and by 4 without remainder; ſuch a one is the number 12, which is the product of 3 and 4 multiplied together; (ſee obſervation the third upon the definition of diviſion:) make then 3 men or 4 women equivalent to 12 boys, and you will have 1 man equivalent to 4 boys, 1 woman to 3 boys, and 1 man and 1 woman to 7 boys, and the queſtion will ſtand thus; *If 12 boys will do a piece of work in 56 days, how long will 7 boys be in doing the ſame?*

Anſwer. 96 days.

QUEST. 38.

If 5 oxen, or 7 colts, will eat up a cloſe in 87 days, in what time will 2 oxen and 3 colts eat up the ſame?

Anſwer. In 105 days.

QUEST. 39.

If 2 acres of land will maintain 3 horſes 4 days, how long will 5 acres maintain 6 horſes?

This queſtion may perhaps at firſt ſight, be taken to be ſomewhat of the ſame nature with the 32d and 33d queſtions, which belonged to the double rule of three direct; but when it comes to be examined into more narrowly, it will be found to be of a very different nature: for we cannot ſay here as we did there, that 2 acres will laſt 3 horſes as long as 1 acre will laſt 6 horſes; this would be a very unjuſt way of thinking, and wherever it is ſo, the queſtion ought to be referred to another rule, which they call the double rule of three inverſe; the propriety or impropriety of this thought, being an infallible criterion whereby to diſtinguiſh, when a queſtion belongs to one rule, and when to the other. All queſtions belonging to this rule, as well as thoſe belonging to the other, may be reduced to the ſingle rule of three two ways; either by expunging the firſt and fourth numbers, or the ſecond and fifth; but then the methods of

of extermination are different. In questions of this nature, if the first and fourth numbers are to be expunged, the 2 first numbers are to be multiplied by the fourth, and the 2 last by the first; but if the second and fifth numbers are to be expunged, then the two first numbers are to be multiplied by the fifth, and the two last by the second: thus in the question before us, if we would exterminate the first and fourth numbers, we must multiply the two first numbers, that is, 2 and 3, by the fourth, that is, by 5, and say, that 2 acres will last 3 horses just as long as 10 acres will last 15 horses; we must also multiply the 2 last numbers, to wit, 5 and 6, by the first, that is, by 2, and say, that 5 acres will last 6 horses as long as 10 acres will last 12 horses: use now these numbers instead of those in the question, and it will be changed into this equivalent one; If 10 acres of land will maintain 15 horses 4 days, how long will 10 acres maintain 12 horses? Strike out of the question the first and fourth numbers, which being equal, will be of no use in the conclusion, and then the question will stand thus; *If 15 horses will eat up a certain piece of ground in 4 days, how long will 12 horses be in eating up the same?*

Answer. 5 days; for this question belongs to the rule of three inverse.

If we would exterminate the second and fifth numbers out of the question, we must multiply the two first numbers by the fifth, and say, that 2 acres will last 3 horses just as long as 12 acres will last 18 horses; we must also multiply the 2 last numbers by the second, and say, that 5 acres will last 6 horses as long as 15 acres will last 18 horses: use these numbers instead of those in the question, and it will be changed into this equivalent one; If 12 acres will maintain 18 horses 4 days, how long will 15 acres maintain 18 horses? That is, (striking out the second and fifth numbers) *If 12 acres of land will maintain a certain number of horses 4 days, how long will 15 acres last the same number?*

Answer. 5 days, as before; for this question belongs to the rule of three direct.

In both these operations, the number sought was at last found by multiplying 15 by 4, and then dividing the product by 12: now whosoever looks back upon the foregoing resolution, and observes how these numbers were formed, he will easily perceive, that the number 4 was the middle term in the question; that the number 15 in both operations was the product of the numbers 3 and 5, which lay next the middle term on each side; and that the divisor 12 was in both cases the product of the extreme numbers 2 and 6: therefore *In all questions be-*

longing to the double rule of three inverse, where the numbers are supposed to be ordered as in the double rule of three direct, if the three middle numbers be multiplied together, and the product be divided by the product of the two extremes, the quotient of this division will be the number sought. And thus may all the trouble of expunging be avoided, though I thought it proper to explain that method in the first place, in order to let the learner into the reason of this last theorem which is founded upon it.

Questions wherein the extraction of the square root is concerned.

QUEST. 40.

There is a certain field whose breadth is 576 yards, and whose length is 1296 yards: I demand the side of a square field equal to it.

Answer. This field will be equal to a square whose side is 864 yards.

QUEST. 41.

There is a certain inclosure 3 times as long as it is broad, whose area is 46128 square yards: I demand its breadth and length.

The breadth multiplied into the length, that is, the breadth multiplied into 3 times itself, is 46128; therefore the breadth multiplied into itself is 15376; therefore the breadth is 124, and the length 372.

QUEST. 42.

A certain society collect among themselves a sum amounting to 15 pounds, 5 shillings and a farthing, every one contributing as many farthings as there were members in the whole society: I demand the number of members.

Answer. 121 members.

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INTRODUCTION,

Concerning Vulgar and Decimal Fractions.

DEFINITIONS.

Art. 1. A FRACTION ſimply and abſtractedly conſidered, is that wherein ſome part or parts of an unit are expreſſed: as if an unit be ſuppoſed to be divided into 4 equal parts, and three of theſe parts are to be expreſſed, it muſt be done by the fraction three fourths, to be written thus $\frac{3}{4}$: here the number 4, which ſhews into how many equal parts the unit is ſuppoſed to be divided, and ſo determines the true value, magnitude, or denomination of thoſe parts, is called the denominator of the fraction; and the number 3, which ſhews how many of theſe parts are conſidered in the fraction, is called the numerator: thus in the fraction $\frac{1}{2}$ or one half, 1 is the numerator, and 2 the denominator; in $\frac{2}{2}$ or two halves, 2 is both numerator and denominator, &c.

When a fraction is applied to any particular quantity, that quantity is called the integer to the fraction: thus in $\frac{3}{4}$ of a penny, a penny is the integer; in three fourths of ſix, the number 6 is the integer; thus in three fourths of five ſixths, the fraction five ſixths is the integer; for though in an abſolute ſenſe it be a fraction, yet here with reſpect to the fraction three fourths, it is an integer: and thus may one and the ſame quantity, under different ways of conception, be both an integer and a fraction; as a foot is an integer, and a third part of a yard is a fraction, though they both ſignify the ſame thing. When the integer to a fraction is not expreſſed, unity is always to be underſtood: thus $\frac{3}{4}$ is $\frac{3}{4}$ of an unit; thus when we ſay, $\frac{1}{3}$ and $\frac{1}{4}$ make $\frac{7}{12}$, the meaning is, that if $\frac{1}{3}$ part of an unit, and $\frac{1}{4}$ part of an unit be added together, the ſum will amount to the ſame as if that unit had been divided into 12 equal parts, and 7 of thoſe parts had been taken; thus again, when we ſay that $\frac{2}{3}$ of $\frac{4}{5}$ are equivalent to $\frac{8}{15}$, we mean, that if an unit be divided into 5 equal parts, and 4 of them be taken, and then this fraction $\frac{4}{5}$ be again divided into 3 equal

parts, and 2 of them be taken, the result will be the same, as if the unit had at first been divided into 15 equal parts, and 8 of them had been taken; and whatever is true in the case of unity, will be equally true in the case of any other integer whatever: thus if it be true that $\frac{1}{3}$ and $\frac{1}{4}$ of an unit are equal to $\frac{7}{12}$ of an unit, that is, if it be true in general that $\frac{1}{3}$ and $\frac{1}{4}$ added together are equal to $\frac{7}{12}$, it will be as true of any particular integer, suppose of a pound sterling, that $\frac{1}{3}$ of a pound, and $\frac{1}{4}$ of a pound when added together, are equal to $\frac{7}{12}$ of a pound; again, if it be true in general that $\frac{2}{3}$ of $\frac{4}{5}$ are equal to $\frac{8}{15}$, it is as true in particular that $\frac{2}{3}$ of $\frac{4}{5}$ of a pound are equivalent to $\frac{8}{15}$ of a pound, &c.

Of proper and improper fractions; and of the reduction of an improper fraction to a whole or mixt number.

2. Fractions are of two sorts, proper and improper; a proper fraction is that, whose numerator is less than the denominator, as $\frac{1}{2}$; therefore an improper one is that, whose numerator is equal to, or greater than the denominator, as $\frac{2}{2}$, $\frac{3}{2}$, &c.

OBJECTION.

But is there no absurdity in the supposition of an improper fraction, as in three halves for instance, considering that an unit cannot be divided into more than two halves? *Answer:* no more than there is in supposing three half pence to be the price of any thing, considering that a penny cannot be divided into above two halfpence. These fractions therefore are called improper, not from any absurdity either in the supposition or in the expression, but because they may be more properly and more intelligibly expressed, either by a whole number, or at least by a mixt number consisting of a whole number and a fraction; as for example, if the numerator of a fraction be equal to the denominator, as $\frac{4}{4}$, that fraction will always be equivalent to unity, as $\frac{4}{4}$ of an hour, that is, four quarters of an hour, are equivalent to one hour, $\frac{4}{4}$ of a penny, that is, four farthings, are equal to one penny, &c: and the reason is plain; for if an unit be divided into four equal parts, and four of these parts be expressed in a fraction, the whole unit is expressed in that fraction, that is, such a fraction must always be looked upon as equal to an unit: therefore if the numerator be double of the denominator, as $\frac{8}{4}$, the fraction must be equal to the number 2, because $\frac{8}{4}$ contain $\frac{4}{4}$ or 1 twice; in like manner $\frac{12}{4}$ are equal to, and may be more properly expressed by, the number 3; $\frac{16}{4}$ by the number 4, &c: and universally, as often as the numerator

numerator of a fraction contains the denominator, ſo many units is that fraction equivalent to: but to find how often the numerator contains the denominator, is to divide the numerator by the denominator; therefore if the numerator of an improper fraction be divided by the denominator, the quotient, if nothing remains, will be the whole number by which the fraction may be expreſſed; but if any thing remains of this diviſion, then the quotient together with a fraction whoſe numerator is that remainder, and denominator the diviſor, will be a mixt number, expreſſing the fraction propoſed: thus $\frac{24}{3}$ are equivalent to the whole number 8, but $\frac{25}{3}$ are equivalent to the mixt number $8\frac{1}{3}$, $\frac{26}{3}$ to the mixt number $8\frac{2}{3}$, juſt as 24 feet are equal to 8 yards, 25 feet to 8 yards and 1 foot, 26 feet to 8 yards and 2 feet, &c: and this is what we call the reduction of an improper fraction into a whole or mixt number.

The reduction of a whole or mixt number into an improper fraction.

3. As unity may be expreſſed by any fraction of any form or denomination whatever, provided the numerator be equal to the denominator, as $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, &c; ſo the number 2 is reducible to any fraction whoſe numerator is double the denominator, as $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, &c; and ſo is every number reducible to any fraction, whoſe numerator contains the denominator as often as there are units in the number propoſed: therefore whenever a whole number is to be reduced to a fraction whoſe denominator is given, it muſt be multiplied by that given denominator, and the product with that denominator under it, will be the equivalent fraction; thus if the number 5 is to be reduced into halves, that is, into a fraction whoſe denominator is 2, it muſt be multiplied by 2, and ſo you will have 5 equal to $\frac{10}{2}$, juſt as 5 pence are equivalent to 10 halfpence; if the number 8 is to be reduced into thirds, it muſt be multiplied by 3, and ſo you will have 8 equal to $\frac{24}{3}$, juſt as 8 yards are equal to 24 feet; laſtly, if the number 2 is to be reduced into fourths, it will be equal to $\frac{8}{4}$, juſt as 2 pence are equal to 8 farthings. If the number to be reduced be a mixt number, conſiſting of a whole number and a fraction, the whole number muſt always be reduced to the ſame denomination with the fraction annexed, and the rule will be this: multiply the whole number by the denominator of the fraction annexed; add the numerator to the product, and the ſum with the denominator under it will be the equivalent fraction: Thus the mixt number $5\frac{1}{2}$ is equivalent to $\frac{11}{2}$, juſt as 5 pence halfpenny in money is equivalent to 11 halfpence: This operation carries it's own evidence along with it; for the number 5 itſelf is equal to $\frac{10}{2}$,

as

as above; therefore $5\frac{1}{2}$ muſt be equivalent to $\frac{11}{2}$: again, the number $8\frac{2}{3}$ is equal to $\frac{26}{3}$, juſt as 8 yards and 2 feet over are equivalent to 26 feet; laſtly, $2\frac{3}{4}$ is reducible to $\frac{11}{4}$, juſt as 2 pence and 3 farthings are reducible to 11 farthings.

A LEMMA.

4. *If any integer be aſſumed, as a pound ſterling, and alſo any fraction, as* $\frac{3}{4}$, *I ſay then, that* $\frac{3}{4}$ *parts of one pound, amount to the ſame as* $\frac{1}{4}$ *part of* 3 *pounds.*

To demonſtrate this Lemma (which ſcarce wants a demonſtration) I argue thus: If any quantity, greater or leſs, be always divided into the ſame number of parts, the greater or leſs the quantity ſo divided is, the greater or leſs will the parts be; thus $\frac{1}{4}$ of a yard is 3 times as much as $\frac{1}{4}$ of a foot, becauſe a yard is 3 times as much as a foot; and for the ſame reaſon $\frac{1}{4}$ of 3 pounds is 3 times as much as $\frac{1}{4}$ of 1 pound; but $\frac{3}{4}$ of 1 pound are alſo 3 times as much as $\frac{1}{4}$ of 1 pound; therefore $\frac{3}{4}$ of 1 pound are equal to $\frac{1}{4}$ of 3 pounds, becauſe both are juſt 3 times as much as $\frac{1}{4}$ of 1 pound. *Q. E. D.*

How to eſtimate any fractional parts of an integer in parts of a leſſer denomination, and vice verſâ.

5 This may be done various ways; but the ſhorteſt and ſafeſt, as I take it, is that which follows: Suppoſe I had a mind to know the value of $\frac{5}{6}$ of a pound; I ſhould argue as in the foregoing lemma, that $\frac{5}{6}$ of one pound, are the ſame as $\frac{1}{6}$ of 5 pounds; but the latter is more eaſily taken than the former; therefore I apply myſelf wholly to the latter, to wit, to find the ſixth part of 5 pounds, thus: 5 pounds, or 100 ſhillings, divided by 6, quote 16 ſhillings, and there remain 4 ſhillings; again, 4 ſhillings, or 48 pence, divided by 6, quote 8 pence, and there remains nothing; therefore the value of 1 ſixth of 5 pounds, or $\frac{5}{6}$ of 1 pound, is 16 ſhillings and 8 pence. Again, ſuppoſe I would know the value of $\frac{6}{7}$ of a pound, I find the value of $\frac{1}{7}$ of 6 pounds thus; 6 pounds, or 120 ſhillings divided by 7, give 17 ſhillings, and there remains 1 ſhilling; again, 1 ſhilling, or 12 pence, divided by 7, give 1 penny, and there remain 5 pence; again, 5 pence, or 20 farthings, divided by 7, give 2 farthings, and there remain 6 farthings; laſtly, a ſeventh part of 6 farthings is juſt as much as $\frac{6}{7}$ of 1 farthing, by the lemma: hence I conclude, that $\frac{6}{7}$ of a pound are 17 ſhillings, 1 penny, 2 farthings, and $\frac{6}{7}$ of a farthing: But the value of $\frac{6}{7}$ of a farthing is ſo near to one farthing, that if I would rather

rather admit of a small inaccuracy in my account, than a fraction, I should make the value of $\frac{6}{7}$ of a pound to be 17 shillings, 1 penny and 3 farthings. Lastly, suppose I would know the amount of $\frac{2}{3}$ parts of 17 shillings and sixpence, I should argue thus; $\frac{2}{3}$ parts of 17 shillings and sixpence, are equivalent to $\frac{1}{3}$ part of twice as much, that is, to $\frac{1}{3}$ part of 35 shillings: but $\frac{1}{3}$ part of 35 shillings is 11 shillings and 8 pence; therefore $\frac{2}{3}$ parts of 17 shillings and sixpence make 11 shillings and 8 pence.

Of the reverse of this reduction, one single instance will suffice: Let it then be required to reduce 1 shilling, 2 pence, 3 farthings, to fractional parts of a pound: here I consider, that in 1 pound are 960 farthings; and in 1 shilling, 2 pence, 3 farthings, are 59 farthings; therefore 1 farthing is $\frac{1}{960}$ of a pound; and 1 shilling, 2 pence, 3 farthings, are $\frac{59}{960}$ of a pound.

Preparations for further reductions and operations of fractions.

6 All the operations and reductions of fractions, are mediately or immediately deducible from the following principle, which is; that *If the numerator of a fraction be encreased, whilst the denominator continues the same, the value of the fraction will be encreased proportionably; and* vice versâ. *On the other hand, if the denominator be encreased in any proportion, whilst the numerator continues the same, the value of the fraction will be diminished in a contrary proportion; and* vice versâ. Thus $\frac{2}{3}$ are twice as much as $\frac{1}{3}$, and $\frac{1}{6}$ is but half as much.

From this principle it follows, that if the numerator and denominator of a fraction be both multiplied, or both divided by the same number, the value of the fraction will not be affected thereby; because, as much as the fraction is encreased by multiplying the numerator, just so much again it will be diminished by multiplying the denominator; and as much as the fraction is diminished by dividing the numerator, just so much again it will be encreased by dividing the denominator: Thus the terms of the fraction $\frac{3}{4}$ being doubled, produce $\frac{6}{8}$, a fraction of the same value; and on the contrary, the terms of the fraction $\frac{6}{8}$ being halved, give $\frac{3}{4}$.

Hence it appears, that every fraction is capable of infinite variety of expression, since there is infinite choice of multiplicators, whereby the numerator and denominator of a fraction may be multiplied, and so the expression may be changed, without changing the value of the fraction: thus the fraction $\frac{1}{2}$, if both the numerator and denominator be multiplied by 2, becomes $\frac{2}{4}$; if by 3, $\frac{3}{6}$; if by 4, $\frac{4}{8}$; if by 5, $\frac{5}{10}$; and so on *ad infinitum*; all which are nothing else but different expressions of the same fraction:

fraction: therefore in the midſt of ſo much variety, we muſt not expect that every fraction we meet with ſhould always be in it's leaſt or loweſt terms; but how to reduce them to this ſtate whenever they happen to be otherwiſe, ſhall be the buſineſs of the next article.

The reduction of fractions from higher to lower terms.

7. Whenever a fraction is ſuſpected not to be in it's leaſt terms, find out, if poſſible, ſome number that will divide both the numerator and denominator of the fraction without any remainder; for if ſuch a number can be found, and the diviſion be made, the two quotients thence ariſing will exhibit reſpectively, the numerator and denominator of a fraction, equal to the fraction firſt propoſed, but expreſſed in more ſimple terms: this is evident from the laſt article. As for example; let the fraction $\frac{10}{15}$ be propoſed to be reduced: here to find ſome number that will divide both the numbers 10 and 15 without any remainder, I begin with the number 2, as being the firſt whole number that can have any effect in diviſion; but I find 2 will not divide 15; 3 is the next number to be tried; but neither will that ſucceed, for it will not divide 10; as for the number 4, I paſs that by, becauſe if 2 would not divide 15, much leſs will 4 do it; the next number I try is 5, and that ſucceeds; for if 10 and 15 be divided by 5, the quotients will be 2 and 3 reſpectively, each without remainder; therefore the fraction $\frac{10}{15}$, after being reduced to it's leaſt terms, is found to be the ſame as $\frac{2}{3}$; that is, if an unit be divided into 15 equal parts, and 10 of them be taken, the amount will be the ſame, as if it had been divided into 3 equal parts, and 2 of them had been taken. Secondly, if the fraction propoſed to be reduced be $\frac{2520}{7560}$, divide it's terms by 2, and you will have the fraction $\frac{1260}{3780}$; divide again by 2, and you will have $\frac{630}{1890}$; divide again by 2, and you will have $\frac{315}{945}$; therefore all further diviſion by 2 is excluded: divide then theſe laſt terms by 3, and you will have $\frac{105}{315}$; divide again by 3, and you will have $\frac{35}{105}$; divide by 5, and you will have $\frac{7}{21}$; and laſtly divide by 7, and you will have $\frac{1}{3}$; ſo that the fraction $\frac{2520}{7560}$, after a common diviſion by 2, 2, 2, 3, 3, 5, 7, is found at laſt equal to $\frac{1}{3}$. Thirdly, the fraction $\frac{36}{48}$, after a continual diviſion by 2, 2, 3, becomes $\frac{3}{4}$. Fourthly, $\frac{56}{84}$ after a continual diviſion by 2, 2, 7, becomes $\frac{2}{3}$. Fifthly, $\frac{144}{180}$ after a conti-

continual division by 2, 2, 3, 3, becomes $\frac{4}{5}$. Sixthly, $\frac{42}{126}$ after a continual division by 2, 3, and 7, becomes $\frac{1}{3}$. Seventhly, $\frac{315}{840}$ after a continual division by 3, 5, 7, becomes $\frac{3}{8}$. Eighthly, $\frac{35}{840}$ after a continual division by 5 and 7, becomes $\frac{1}{24}$. Ninthly, $\frac{735}{245}$ after a continual division by 5, 7, 7, becomes $\frac{3}{1}$, or 3.

Some perhaps may think themselves helped in the practice of this rule by the following observations:

First, that 2 will divide any number that ends with an even number, or with a cypher, as 36, 30, *&c.* and no other.

Secondly, that 5 will divide any number that ends with a 5, or with a cypher, as 75, 70, *&c.* and no other.

Thirdly, that 3 will divide any number, when it will divide the sum of it's digits added together: thus 3 will divide 471, because it will divide the number 12, which is the sum of the numbers 4, 7 and 1.

Fourthly, if both the numerator and denominator have cyphers annexed to them, throw away as many as are common to both: thus $\frac{3500}{56000}$ is the same as $\frac{35}{560}$, or $\frac{7}{112}$, or $\frac{1}{16}$.

After all, there is a certain and infallible rule for finding the greatest common divisor of any two numbers whatever, that have one, whereby a fraction may be reduced to it's least terms by one single operation only. I shall be forced indeed to postpone the demonstration of this rule to a more convenient place, not so much for want of principles to proceed upon, as for want of a proper notation; but the rule itself is as follows: Let a and b be two given numbers, whose greatest common divisor is required; to wit, a the greater, and b the less: then dividing a by b without any regard to the quotient, call the remainder c; divide again b by c, and call the remainder d; then divide c by d, and call the remainder e; then divide d by e, and call the remainder f; and so proceed on, till at last you come to some divisor, as f, which will divide the preceding number e without a remainder: I say then, that this last divisor will be the greatest common divisor of the two given numbers a and b. As for example; let a be 1344 and b 582: then to find the greatest common divisor of these numbers, I divide a (1344) by b (582) and there remains 180, which I call c; then I divide b (582) by c (180) and there remains 42, which I call d; then I divide c (180) by d (42) and there remains 12, which I call e; then I divide d (42) by e (12) and there remains 6, which I call f; lastly I divide e (12) by f (6)

 and

and there remains nothing: whence I conclude that 6 is the greatest common divisor of the two numbers 1344 and 582; and as the quotients by 6 are 224 and 97, it follows, that the fraction $\frac{582}{1344}$, when reduced to it's least terms, will be $\frac{97}{224}$. If no common divisor can be found but unity, it is an argument that the fraction is in it's least terms already.

From this and the last article it follows, that all fractions that are reducible to the same least terms are equal; as $\frac{4}{6}$, $\frac{6}{9}$, $\frac{10}{15}$, *&c*, which are all reducible to $\frac{2}{3}$; though it does not follow *è converso*, that all equal fractions are reducible to the same least terms; this will be demonstrated in another place. (See *Elements of Algebra*, Art. 193.)

For the better understanding of the following article, it must be observed, that this mark × is a sign of multiplication, and is usually read *into*: thus 2 × 3 signifies 6, 2 × 3 × 4 signifies 24, 2 × 3 × 4 × 5 signifies 120, *&c*; and in some cases it will be better to put down these components or factors, than the character of the number arising from their continual multiplication, as in the following article. It ought also to be observed, that it matters not in what order these components are placed; for 2 × 3 × 4 × 5 signifies just the same as 4 × 5 × 2 × 3, *&c*.

The reduction of fractions of different denominations, to others of the same denomination.

8. There is another reduction of fractions, no less useful than the former; and that is, the reduction of fractions of different denominations to others of the same denomination, or which have the same denominator, without changing their values; which is done as follows: Having first put down the fractions to be reduced, in any order, one after another, and beginning with the numerator of the first fraction, multiply it by a continual multiplication, into all the denominators but it's own, and put down the product under that fraction; then multiply in like manner, the numerator of the next fraction into all the denominators but it's own, and put down the product under that fraction; and so proceed on through all the numerators, always taking care to except the denominator of that fraction, whose numerator is multiplied. Then multiplying all the denominators together, put down the product under every one of the products last found, and you will have a new set of fractions, all of the same denomination with one another, and all of the same values with their respective original ones. As for example; let it be proposed to reduce the following fractions to the same denomination, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$: *1st*, The numerator of the first fraction is 1, and the denominators

nators of the reſt are 4, 6 and 8, and $1 \times 4 \times 6 \times 8$ gives 192; therefore I put down 192 under $\frac{1}{2}$. 2*dly*, The numerator of the ſecond fraction is 3, and the denominators of the reſt are 6, 8 and 2, and $3 \times 6 \times 8 \times 2$ gives 288; therefore I put down 288 under $\frac{3}{4}$. 3*dly*, $5 \times 8 \times 2 \times 4$ gives 320; therefore I put down 320 under $\frac{5}{6}$. 4*thly*, $7 \times 2 \times 4 \times 6$ gives 336; therefore I put down 336 under $\frac{7}{8}$. Laſtly, $2 \times 4 \times 6 \times 8$, or the product of all the denominators is 384: this therefore I put down under every one of the numerators laſt found, and ſo have a new ſet of fractions, *viz.* $\frac{192}{384}$, $\frac{288}{384}$, $\frac{320}{384}$, $\frac{336}{384}$, all of the ſame denomination, as appears from the operation itſelf; and all of the ſame value with their reſpective original ones, as will appear preſently; but firſt ſee the work:

$$\frac{1}{2}. \quad \frac{3}{4}. \quad \frac{5}{6}. \quad \frac{7}{8}.$$

$$\frac{192}{384}. \quad \frac{288}{384}. \quad \frac{320}{384}. \quad \frac{336}{384}$$

A demonſtration of the rule.

All that is to be demonſtrated in this rule is, to prove from the nature of the operation itſelf, that the original fractions ſuffer nothing in their values by this reduction: in order to which, it will be convenient to put down the components of the new numerators inſtead of their proper characters, as in the laſt article; as alſo thoſe of the common denominator, and the work will ſtand thus:

$$\frac{1}{2}. \quad \frac{3}{4}. \quad \frac{5}{6}. \quad \frac{7}{8}.$$

$$\frac{1 \times 4 \times 6 \times 8}{2 \times 4 \times 6 \times 8}. \quad \frac{3 \times 6 \times 8 \times 2}{4 \times 6 \times 8 \times 2}. \quad \frac{5 \times 8 \times 2 \times 4}{6 \times 8 \times 2 \times 4}. \quad \frac{7 \times 2 \times 4 \times 6}{8 \times 2 \times 4 \times 6}.$$

By this method of operation it appears, that the numerator and denominator of the firſt fraction $\frac{1}{2}$, are both multiplied by the ſame number in the reduction, to wit, by $4 \times 6 \times 8$; and therefore that fraction ſuffers nothing in it's value, by art. 6. In like manner, the terms of the ſecond fraction $\frac{3}{4}$ are both multiplied by the ſame number $6 \times 8 \times 2$; therefore that fraction can ſuffer nothing in it's value; and the ſame may be ſaid of all the reſt. *Q. E. D.*

Other examples to this rule.

$$\frac{1}{2}. \quad \frac{1}{3}. \quad \frac{1}{4}. \quad \frac{1}{5}. \quad \frac{1}{6}. \qquad\qquad \frac{1}{3}. \quad \frac{1}{4}. \quad \frac{1}{5}. \quad \frac{1}{6}.$$

$$\frac{360}{720}. \quad \frac{240}{720}. \quad \frac{180}{720}. \quad \frac{144}{720}. \quad \frac{120}{720}. \qquad\qquad \frac{120}{360}. \quad \frac{90}{360}. \quad \frac{72}{360}. \quad \frac{60}{360}$$

$\frac{1}{4}$	$\frac{1}{5}$.	$\frac{1}{6}$.	$\frac{1}{5}$.	$\frac{1}{6}$.
$\frac{30}{120}$.	$\frac{24}{120}$.	$\frac{20}{120}$.	$\frac{6}{30}$.	$\frac{5}{30}$

The uſe of this rule will ſoon appear in the addition and ſubtraction of fractions: in the mean time it may not be amiſs to obſerve, that it would be very difficult, if not impoſſible, to compare fractions of different denominations, without firſt reducing them to the ſame. As for inſtance; ſuppoſe it ſhould be asked, which of theſe two fractions is the greater, $\frac{3}{4}$, or $\frac{5}{7}$; in this view it would be difficult to determine the queſtion; but when I know that $\frac{3}{4}$ are the ſame with $\frac{21}{28}$, and that $\frac{5}{7}$ are the ſame with $\frac{20}{28}$, I know then, that $\frac{3}{4}$ are greater than $\frac{5}{7}$ by a twenty eighth part of the whole. We now proceed to the four operations of fractions, to wit, their addition, ſubtraction, multiplication and diviſion: and firſt,

Of the addition of fractions.

9. Whenever two or more fractions are to be added together, let them firſt be reduced to the ſame denomination, if they be not ſo already; and then adding the new numerators together, put down the ſum with the common denominator under it. In the caſe of mixt numbers, add firſt the fractions together, and then the whole numbers: but if the fractions when added together, make an improper fraction, reduce it by the 2d art. to a whole, or mixt number; and then putting down the fractional part, if there be any, reſerve the whole number for the place of integers.

To this rule might be referred (if it had not been taught already in the 3d art.) the reduction of a mixt number into an improper fraction, which is nothing elſe but adding a whole number and a fraction together, and may be done by conſidering the whole number as a fraction whoſe denominator is unity.

Examples of addition in fractions.

1ſt, $\frac{3}{10}$ and $\frac{4}{10}$ when added together make $\frac{7}{10}$, for juſt the ſame reaſon as 3 ſhillings and 4 ſhillings when added together make 7 ſhillings.

2dly, The fractions $\frac{1}{3}$ and $\frac{1}{4}$ when reduced to the ſame denomination by the laſt art. are $\frac{4}{12}$ and $\frac{3}{12}$, and theſe added together make $\frac{7}{12}$; therefore the fractions $\frac{1}{3}$ and $\frac{1}{4}$ when added together make up the fraction $\frac{7}{12}$.

For a better confirmation of theſe abſtract concluſions, but chiefly to inure the learner to conceive and reaſon diſtinctly about fractions, it will be

be very convenient to apply these examples in some particular case, as for instance, in the case of a pound sterling; and if we do so here, we are to try, whether $\frac{1}{3}$ and $\frac{1}{4}$ of a pound when added together, amount to $\frac{7}{12}$ of a pound, or not: here then we shall find by division, that the third part of a pound is 6 shillings and 8 pence, and the fourth part 5 shillings; and these added together, make 11 shillings and 8 pence; therefore $\frac{1}{3}$ and $\frac{1}{4}$ of a pound when added together, make 11 shillings and 8 pence; but by the 5th art. it will be found that $\frac{7}{12}$ of a pound are also 11 shillings and 8 pence; therefore $\frac{1}{3}$ and $\frac{1}{4}$ of a pound, when added together, make $\frac{7}{12}$ of a pound; and the same would have been true in any other instance whatever.

3dly, $\frac{2}{5}$ and $\frac{3}{8}$, that is, $\frac{16}{40}$ and $\frac{15}{40}$, when added together, make $\frac{31}{40}$, which will also be true in the case of a pound sterling; for by the 5th art. $\frac{2}{5}$ of a pound are 8 shillings, $\frac{3}{8}$ of a pound are 7 shillings and 6 pence, and their sum is 15 shillings and 6 pence; which will also be found to be the value of $\frac{31}{40}$ of a pound; therefore $\frac{2}{5}$ and $\frac{3}{8}$ of a pound when added together, make $\frac{31}{40}$ of a pound.

4thly, $\frac{2}{3}$ and $\frac{4}{5}$, that is, $\frac{10}{15}$ and $\frac{12}{15}$, when added together, make $\frac{22}{15}$, an improper fraction; which being reduced to a mixt number, by the 2d art. is 1 and $\frac{7}{15}$: let us now try, whether $\frac{2}{3}$ of a pound, and $\frac{4}{5}$ of a pound when added together will make one pound and $\frac{7}{15}$ of a pound over, or not: now $\frac{2}{3}$ of a pound, or 13 shillings and 4 pence, added to $\frac{4}{5}$ of a pound, or 16 shillings, amount to 1 pound 9 shillings and 4 pence; and $\frac{7}{15}$ of a pound, are found to be 9 shillings and 4 pence; therefore $\frac{2}{3}$ and $\frac{4}{5}$ of a pound when added together, make one pound and $\frac{7}{15}$ of a pound over.

5thly, $\frac{3}{4}$ and $\frac{5}{6}$, that is, $\frac{18}{24}$ and $\frac{20}{24}$: when added together, make $\frac{38}{24}$, or $1\frac{7}{12}$, which will also be true in the case of a pound sterling.

6thly, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, that is, $\frac{360}{720}$, $\frac{240}{720}$, $\frac{180}{720}$, $\frac{144}{720}$, $\frac{120}{720}$, when added together, make $\frac{1044}{720}$; that is, $1\frac{9}{20}$: try it in money.

7thly, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ and $\frac{5}{6}$, that is, $\frac{360}{720}$, $\frac{480}{720}$, $\frac{540}{720}$, $\frac{576}{720}$ and $\frac{600}{720}$, when added together, make $\frac{2556}{720}$, that is, $3\frac{11}{20}$.

8thly, The sum of the mixt numbers $7\frac{1}{3}$ and $8\frac{1}{4}$ is $15\frac{7}{12}$; for the sum of the fractions is $\frac{7}{12}$ by the second example, and the sum of the whole numbers is 15.

9thly, $5\frac{2}{3}$ added to $7\frac{4}{5}$ gives $13\frac{7}{15}$; for the sum of the fractions is $1\frac{7}{15}$, by the fourth example; and the whole number 1, added to the whole numbers 5 and 7 gives 13.

10thly,

10*thly*, $8\frac{1}{2}$, $9\frac{2}{3}$, $10\frac{3}{4}$, $11\frac{4}{5}$, $12\frac{5}{8}$ added together make $53\frac{11}{20}$; for the fractions themſelves make $3\frac{11}{20}$ by the ſeventh example, and the whole number 3 added to the reſt makes 53.

11*thly*, The whole number 2 added to the fraction $\frac{3}{4}$ gives $\frac{11}{4}$; for the whole number 2 may be conſidered as a fraction, whoſe denominator is unity; now $\frac{2}{1}$ and $\frac{3}{4}$ when reduced to the ſame denomination, are $\frac{8}{4}$ and $\frac{3}{4}$, which added together make $\frac{11}{4}$.

•Thus alſo may unity be added to any fraction whatever, when ſubtraction requires it; but better thus: unity may be made a fraction of any denomination whatever, provided the numerator be equal to the denominator, by art. 2d: ſuppoſe then I would add unity to $\frac{2}{3}$; I ſuppoſe unity equal to $\frac{3}{3}$, and this added to $\frac{2}{3}$ makes $\frac{5}{3}$: again, unity added to $\frac{3}{5}$ makes $\frac{8}{5}$, becauſe $\frac{5}{5}$ and $\frac{3}{5}$ make $\frac{8}{5}$.

Of the ſubtraction of fractions.

10. Whenever a leſs fraction is to be ſubtracted from a greater, they muſt be prepared as in addition; that is, they muſt be reduced to the ſame denomination, if they be not ſo already; then ſubtracting the numerator of the leſs fraction from that of the greater, put down the remainder with the common denominator under it. In the caſe of mixt numbers, ſubtract firſt the fraction of the leſſer number from that of the greater, and then the leſſer whole number from the greater: but if, as it often happens, the greater number has the leſſer fraction belonging to it, then an unit muſt be borrowed from the whole number and added to the fraction, as intimated in the cloſe of the laſt article.

Examples of ſubtraction in fractions.

1*ſt*, $\frac{3}{10}$ ſubtracted from $\frac{4}{10}$ leaves $\frac{1}{10}$, juſt in the ſame manner as 3 ſhillings ſubtracted from 4 ſhillings leaves 1 ſhilling.

2*dly*, $\frac{3}{4}$ ſubtracted from $\frac{5}{6}$, that is, $\frac{18}{24}$ ſubtracted from $\frac{20}{24}$, leaves $\frac{2}{24}$, or $\frac{1}{12}$. So $\frac{3}{4}$ of a pound, or 15 ſhillings, ſubtracted from $\frac{5}{6}$ of a pound, or 16 ſhillings and 8 pence, leaves $\frac{1}{12}$ of a pound, that is, 1 ſhilling and 8 pence.

3*dly*, $7\frac{1}{3}$ ſubtracted from $8\frac{1}{2}$, that is, $7\frac{2}{6}$ ſubtracted from $8\frac{3}{6}$, leaves $1\frac{1}{6}$.

4*thly*, $7\frac{3}{4}$ ſubtracted from $8\frac{1}{4}$, that is, $7\frac{3}{4}$ ſubtracted from $7\frac{5}{4}$, leaves $\frac{2}{4}$, or $\frac{1}{2}$; for here the greater number having the leſs fraction belonging to it, I borrow an unit from the whole number 8, and ſo reduce it to 7; and then this unit, under the name of $\frac{4}{4}$, I add to the fraction $\frac{1}{4}$, and ſo make it $\frac{5}{4}$.

5*thly*,

5thly, $7\frac{2}{3}$ ſubtracted from $8\frac{1}{2}$, that is, $7\frac{4}{6}$ ſubtracted from $8\frac{3}{6}$, that is, $7\frac{4}{6}$ ſubtracted from $7\frac{9}{6}$, leaves $\frac{5}{6}$.

6thly, $7\frac{3}{5}$ ſubtracted from 8, that is, $7\frac{3}{5}$ ſubtracted from $7\frac{5}{5}$, leaves $\frac{2}{5}$.

Of the multiplication of fractions.

11. To multiply a whole number is to take the multiplicand as often as that whole number expreſſes: therefore to multiply by a mixt number, is, not only to take the multiplicand as often as the integral part expreſſes, but alſo to take ſuch a part or parts of it over and above, as is expreſſed by the fraction annexed. Thus 10 multiplied by $2\frac{1}{2}$, produces 25: for as $2\frac{1}{2}$ is a middle number between 2 and 3, ſo the product ought to be a middle number between 20 and 30, that is, 25: In like manner 10 multiplied by $1\frac{1}{2}$ produces 15, and being multiplied by $\frac{1}{2}$ produces 5: therefore to multiply by a proper fraction, is nothing elſe but to take ſuch a part or parts of the multiplicand, as is expreſſed by that fraction. Certainly to take 10 twice and half of it over, once and half of it over, no times and half of it over, (which laſt is taking the half of 10,) are operations of the ſame kind, and differ only in degree one from another; and therefore, if the two former operations paſs by the name of multiplication, this laſt ought to do ſo too; and if there be any abſurdity in the caſe, it lies in the name, and not in the thing.

Arithmetic was at firſt employed about whole numbers only, and thus far the name of multiplication was adequate enough, except in the caſe of unity. But it being afterwards conſidered, that no quantity whatever could be called an unit, that was not further diviſible; and conſequently, that there was not only an infinity of fractional numbers below unity, but alſo an infinity of mixt numbers between any two whole numbers whatever; it was judged, rightly enough, that the art of Arithmetic would not be perfect till it's operations extended themſelves to this ſort of number alſo; and this being done without changing their names, it was then that the name of multiplication became too ſcanty for the thing ſignified: this therefore ought to be attributed to the unavoidable want of foreſight in the firſt impoſers, and not to any imperfection in the ſcience itſelf. This is no more than the caſe of many other arts and ſciences that have outgrown their names. Thus Geometry, that originally and properly ſignified no more than the art of ſurveying, is now defined to be a ſcience treating of the nature and properties of all figures, or rather of the different modifications of extenſion and ſpace; ſo that now ſurveying is the leaſt and loweſt part of that ſcience. Thus Hydroſtatics, which originally ſignified no more than the art of weighing bodies in water, or rather the art of finding out the ſpecific gravities of bodies

bodies by weighing them in water, is now made the name of a ſcience, which treats of the nature and properties of fluids in general; and the ſeveral properties of air and mercury, ſo far as they are fluids, fall under the conſideration of Hydroſtatics, as properly as thoſe of water.

But perhaps it may be further urged, that to take the half of any quantity, is not to multiply, but to divide it. To which I anſwer: that it is impoſſible to take the half of any quantity without dividing it by 2; and conſequently, that to multiply by $\frac{1}{2}$ has the ſame effect as to divide by 2; but this does not prove that multiplication is the ſame as diviſion, but only that theſe two operations, how contrary ſoever, may be made to do each others buſineſs, which is no myſtery to any one who is the leaſt converſant in Arithmetic, and will be further explained in the next article.

A fraction may be multiplied by a whole number two ways; either by multiplying the numerator by that number, or elſe by dividing the denominator by the ſame, where ſuch a diviſion is poſſible: thus if the fraction $\frac{5}{6}$ be to be multiplied by 2, the product will either be $\frac{10}{6}$ by doubling the numerator, or $\frac{5}{3}$ by halving the denominator: this is evident from the 6th art. becauſe a fraction will be equally encreaſed, whether it be by encreaſing the numerator, or by diminiſhing the denominator.

If a fraction be to be multiplied by a fraction, multiply the numerator and denominator of the multiplicand, by the numerator and denominator of the multiplicator reſpectively, and the fraction thence ariſing will be the product ſought: thus if it was required to multiply $\frac{4}{5}$ by $\frac{2}{3}$, or (which amounts to the ſame thing) if it was required to determine how much is $\frac{2}{3}$ of $\frac{4}{5}$, the anſwer would be $\frac{8}{15}$; and the reaſon is plain; for $\frac{1}{3}$ of $\frac{4}{5}$ is $\frac{4}{15}$, by the ſixth art. becauſe making the denominator three times greater, makes the fraction three times leſs; but if $\frac{1}{3}$ of $\frac{4}{5}$ be $\frac{4}{15}$, then $\frac{2}{3}$ of $\frac{4}{5}$ ought to be twice as much, that is, $\frac{8}{15}$; therefore to determine the amount of $\frac{2}{3}$ of $\frac{4}{5}$, the numerator and denominator of $\frac{4}{5}$ muſt be multiplied reſpectively by the numerator and denominator of $\frac{2}{3}$; and the ſame reaſon will hold good in all other inſtances.

If a whole number is to be multiplied by a fraction, either change the multiplicator and multiplicand one for another, and then proceed as above directed; or elſe conſider the multiplicand as a fraction whoſe denominator is unity, and ſo proceed according to the rule for multiplying one fraction by another; by which means both rules will be contracted into one. Thus 6, or $\frac{6}{1}$, multiplied into $\frac{2}{3}$, produces $\frac{12}{3}$, or 4.

If the multiplicator, or multiplicand, or both, be mixt numbers, they muſt firſt be reduced to improper fractions by the third art. and then be multiplied according to the general rule.

Examples

Examples of multiplication in fractions.

1ſt, $\frac{2}{3}$ of $\frac{7}{8}$, multiplying numerators together, and denominators together, is $\frac{14}{24}$, or $\frac{7}{12}$: and ſo we find it in any particular caſe: for $\frac{7}{8}$ of a pound are 17 ſhillings and 6 pence; and $\frac{2}{3}$ of 17 ſhillings and 6 pence, that is, (by the 5th art.) $\frac{1}{3}$ of 35 ſhillings, is 11 ſhillings and 8 pence; therefore $\frac{2}{3}$ of $\frac{7}{8}$ of a pound are 11 ſhillings and 8 pence, which will alſo be found to be the value of $\frac{7}{12}$ of a pound.

Here we may obſerve once for all, that whenever two fractions are to be multiplied together, the product will be the ſame, which ſoever it is that multiplies the other, juſt as it is in whole numbers, and for the ſame reaſon; for if $\frac{7}{8}$ be to be multiplied by $\frac{2}{3}$, then the numbers 7 and 8 muſt be reſpectively multiplied by 2 and 3; but if $\frac{2}{3}$ is to be multiplied by $\frac{7}{8}$, then the numbers 2 and 3 muſt be reſpectively multiplied by 7 and 8, which amounts to the ſame thing; whence it follows, that $\frac{2}{3}$ of $\frac{7}{8}$ come to the ſame as $\frac{7}{8}$ of $\frac{2}{3}$: to confirm this, we have ſeen already that $\frac{2}{3}$ of $\frac{7}{8}$ of a pound amount to 11 ſhillings and 8 pence; let us in the next place enquire into the value of $\frac{7}{8}$ of $\frac{2}{3}$ of a pound: now $\frac{2}{3}$ of a pound are 13 ſhillings and 4 pence; and $\frac{7}{8}$ of 13 ſhillings and 4 pence, that is, $\frac{1}{8}$ of 93 ſhillings and 4 pence, is 11 ſhillings and 8 pence; therefore $\frac{2}{3}$ of $\frac{7}{8}$ of a pound are the ſame as $\frac{7}{8}$ of $\frac{2}{3}$ of a pound, ſince both amount to 11 ſhillings and 8 pence.

2dly, $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{9}{10}$ are $\frac{90}{180}$, or $\frac{1}{2}$: for $2 \times 5 \times 9$ make 90, and $3 \times 6 \times 10$ make 180: thus $\frac{9}{10}$ of a pound are 18 ſhillings; and $\frac{5}{6}$ of 18 ſhillings are 15 ſhillings; and $\frac{2}{3}$ of 15 ſhillings are 10 ſhillings; which are $\frac{1}{2}$ of a pound.

3dly, $\frac{3}{4}$ of $\frac{3}{4}$ of $\frac{3}{4}$ are $\frac{27}{64}$: thus $\frac{3}{4}$ a pound are 15 ſhillings; and $\frac{3}{4}$ of 15 ſhillings are 11 ſhillings and 3 pence; and $\frac{3}{4}$ of 11 ſhillings and 3 pence are 8 ſhillings and 5 pence farthing; which will alſo be found to be the value of $\frac{27}{64}$ of a pound.

4thly, The mixt number $6\frac{3}{4}$ multiplied by the whole number 7, or the whole number 7 multiplied by the mixt number $6\frac{3}{4}$, will produce in either caſe $47\frac{1}{4}$: for the mixt number $6\frac{3}{4}$ being reduced (by the 3d art.) to an improper fraction, becomes $\frac{27}{4}$; which being multiplied by 7, or $\frac{7}{1}$, makes $\frac{189}{4}$, or, when reduced to a mixt number, $47\frac{1}{4}$.

This multiplication may alſo be made another way, thus: $\frac{3}{4}$ multiplied by 7 makes $\frac{21}{4}$, that is, (by the 2d art.) $5\frac{1}{4}$; put down the fraction $\frac{1}{4}$ and keep the 5 in reſerve; then 6 multiplied by 7 makes 42, which,

with the 5 in reſerve, makes 47; therefore the whole product is $47\frac{1}{4}$ as before.

5thly, $3\frac{3}{4}$ multiplied by $2\frac{2}{3}$, that is, $\frac{15}{4}$ multiplied by $\frac{8}{3}$, makes $\frac{120}{12}$, that is, 10: thus $3\frac{3}{4}$ of a pound are 3 pounds, 15 ſhillings; and twice 3 pounds, 15 ſhillings is 7 pounds, 10 ſhillings; moreover $\frac{2}{3}$ of 3 pounds, 15 ſhillings, or $\frac{1}{3}$ of 7 pounds, 10 ſhillings, is 2 pounds, 10 ſhillings; and theſe 2 pounds, 10 ſhillings added to the former part of the product, to wit, 7 pounds, 10 ſhillings, give 10 pounds for the whole product; therefore $3\frac{3}{4}$ of a pound multiplied by $2\frac{2}{3}$ make 10 pounds.

6thly, $96\frac{1}{2}$ multiplied by $24\frac{1}{3}$, that is, $\frac{193}{2}$ multiplied by $\frac{73}{3}$, gives $\frac{14089}{6}$, that is, (by the 2d art.) $2348\frac{1}{6}$.

7thly, $36\frac{1}{4}$ multiplied into itſelf, that is, $\frac{145}{4}$ multiplied by $\frac{145}{4}$ makes $\frac{21025}{16}$, that is, $1314\frac{1}{16}$.

Before I put an end to this art. I do not know whether it will be thought worth my while to take notice of a very abſurd queſtion ſometimes bandied about, wherein it is required to multiply $\frac{1}{3}$ of a pound by $\frac{1}{2}$ of a pound: I call this a very abſurd queſtion, becauſe there is no manner of propriety in it; for in the very idea and definition of multiplication, the multiplicator at leaſt is ſuppoſed to be an abſtract number, or fraction, otherwiſe, what can be the meaning of taking the multiplicand as often, or as much of it, as is expreſſed by the multiplicator? If by multiplying $\frac{1}{3}$ of a pound by $\frac{1}{2}$ of a pound, be meant no more, than multiplying $\frac{1}{3}$ of a pound by $\frac{1}{2}$, why is the word pound expreſſed in the multiplicator? and if there be any other meaning in it, why does not the propoſer explain it, ſince it is not expreſſed in the queſtion? Let him tell me what he means by multiplying 1 pound by 1 pound, and I will ſoon undertake to anſwer his queſtion; but if he neither can nor will do this, the queſtion neither deſerves, nor is capable of an anſwer. I am not ignorant of another queſtion more frequently uſed than this, and of equal nonſenſe, if cuſtom had not explained it, and that is, to multiply 3 yards by 2 yards, and the like; whereby is meant I ſuppoſe, to aſſign the number of ſquare yards contained in a rectangled parallelogram, or long ſquare, 3 yards in length, and 2 yards in breadth; but if this be the ſenſe put upon that queſtion by common conſent, that is all the title it has to it, there being no ſuch thing either expreſſed, or ſo much as implied, in the terms of the queſtion..

A LEMMA.

12. *Let* n *be any whole number, mixt number, or fraction; I say then that the quotient of* n *divided by any fraction is equal to the product of* n *multiplied into the reverse of that fraction:* as for instance,

Let n be divided by $\frac{3}{2}$; I say that the quotient of n divided by $\frac{3}{2}$, will be equal to the product of n multiplied by $\frac{2}{3}$: for let q be the quotient of n divided by $\frac{3}{2}$; that is, let q be a number expressing how often the fraction $\frac{3}{2}$ is contained in n; then will $\frac{3}{2}$ multiplied by q, be equal to n, from the nature of multiplication; but the product of $\frac{3}{2}$ multiplied by q is the same with the product of q multiplied by $\frac{3}{2}$; that is, $\frac{3}{2}$ of q, by the last article; therefore n is equal to $\frac{3}{2}$ of q; therefore $\frac{1}{3}$ of n is equal to $\frac{1}{2}$ of q; therefore $\frac{2}{3}$ of n are equal to q; but $\frac{2}{3}$ of n is the product of n multiplied by $\frac{2}{3}$; therefore the product of n multiplied by $\frac{2}{3}$ is equal to q; but the quotient of n divided by $\frac{3}{2}$ was q, by the supposition; therefore the quotient of n divided by $\frac{3}{2}$, is equal to the product of n multiplied by $\frac{2}{3}$. *Q. E. D.*

COROLLARY.

Hence may the rule of division be at any time changed into that of multiplication, only by inverting the terms of the divisor, and then multiplying instead of dividing. The same will also obtain in whole numbers, if they be considered as fractions whose denominators are units: thus to divide n by 2, that is, $\frac{2}{1}$, will have the same effect as to multiply it by $\frac{1}{2}$, as was hinted in the foregoing article.

Of the division of fractions.

13. The division of fractions like all other division, is to find how often one fraction called the divisor, is contained in another called the dividend; and that which shews this, is called the quotient, whether it be a whole number, a mixt number, or a proper fraction: for in fractional division the quotient is always intended to be exact, without any remainder. and therefore must sometimes be a whole number, sometimes a mixt number, and sometimes a proper fraction. Thus if 18 is to be divided by 6, the quotient will be 3; because 18 contains 6 3 times: but if 21 is to be divided by 6, the quotient will be $3\frac{1}{2}$; because 21 contains 6 three times, and half of it over and above: lastly, if 3 is to be divided by 6, the quotient will be $\frac{1}{2}$; because here the divisor being greater

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than the dividend, cannot be so much as once contained in it, and therefore the quotient in this case, must be a proper fraction, that is, $\frac{1}{2}$, since 3 is just the half of 6.

A fraction may be divided by a whole number two ways; either by dividing the numerator by that whole number when possible, or else by multiplying the denominator by the same: thus the half of $\frac{6}{7}$ may be taken, that is, $\frac{6}{7}$ may be divided by 2, either by halving the numerator, and the quotient will be $\frac{3}{7}$, or else by doubling the denominator, and then the quotient will be $\frac{6}{14}$, both which amount to the same thing, by the 6th and 7th articles.

If the divisor be a fraction, the quotient may be had by multiplying the dividend into the inverted divisor, according to the rules of multiplication already laid down: thus if $\frac{4}{5}$ is to be divided by $\frac{2}{3}$, the quotient will be the same as the product of $\frac{4}{5}$ multiplied by $\frac{3}{2}$, that is, $\frac{12}{10}$, or $1\frac{1}{5}$; the demonstration whereof is contained in the last article.

And here again, as well as in the eleventh article, we are to observe, that if either the divisor, or dividend, or both, be mixt numbers, they must be reduced to improper fractions before the general rule can have place; and that if either, or both be whole numbers, they must be considered as fractions whose denominators are units.

From the general rule of division before laid down it follows, that every fraction may be considered as the quotient of the numerator divided by the denominator, and that, whether the terms of the fraction under consideration be whole numbers, or (which sometimes happens) mixt numbers, or even pure fractions: a demonstration of this last case will serve for all, since mixt numbers may be reduced to fractions, and whole numbers may be considered as fractions whose denominators are units. Let the fraction proposed be $\frac{\frac{4}{5}}{\frac{2}{3}}$; I say, that this fraction is equal to the quotient arising from the division of the numerator $\frac{4}{5}$ by the denominator $\frac{2}{3}$: to demonstrate which, multiply both $\frac{4}{5}$ the numerator, and $\frac{2}{3}$ the denominator, by $\frac{3}{2}$ the inverted denominator, and the fraction will be changed into this, $\frac{\frac{12}{10}}{1}$, or $\frac{12}{10}$, being of the same value with the former, by the 6th art. but the quotient of $\frac{4}{5}$ divided by $\frac{2}{3}$ is also $\frac{12}{10}$ as above; therefore the fraction $\frac{\frac{4}{5}}{\frac{2}{3}}$ is equal to the quotient arising from the division of the numerator by the denominator: and the same way of reasoning may be used in any other instance. This consideration is of very great use in Algebra, where quantities are very often so generally expressed, that there is no other way of representing the quotient, but by a fraction whose numerator is the dividend, and denominator the divisor. Hence also we are taught

how

how to reduce a complicated fraction into a simple one, whose numerator and denominator are whole numbers, to wit, by dividing the numerator by the denominator: thus we see that $\frac{\frac{4}{5}}{\frac{2}{3}}$ is the same as $\frac{12}{10}$.

Other examples of division in fractions.

1st, $\frac{5}{6}$ divided by $\frac{3}{4}$, or, which is the same thing, $\frac{5}{6}$ multiplied into $\frac{4}{3}$, makes $\frac{20}{18}$, or $1\frac{1}{9}$; which shews, that $\frac{3}{4}$ is contained once, and $\frac{1}{9}$ part of it over and above, in $\frac{5}{6}$: for a further confirmation of this, $\frac{5}{6}$ of a pound are 16 shillings and 8 pence; and $\frac{3}{4}$ of a pound are 15 shillings: now 15 shillings are once contained in 16 shillings and 8 pence, and there is 1 shilling and 8 pence over; which 1 shilling and 8 pence is just $\frac{1}{9}$ of 15 shillings. To prevent oversights, the learner is to remember, that it is the terms of the divisor only that are to be inverted, and not those of the dividend: thus to divide $\frac{5}{6}$ by $\frac{3}{4}$ is the same as to multiply $\frac{5}{6}$ into $\frac{4}{3}$, but not the same as to multiply $\frac{6}{5}$ into $\frac{4}{3}$.

2dly, $\frac{9}{10}$ divided by $\frac{1}{3}$, or multiplied into $\frac{3}{1}$, make $\frac{27}{10}$, or $2\frac{7}{10}$, which may be confirmed like the former: for $\frac{9}{10}$ of a pound are 18 shillings; and $\frac{1}{3}$ of a pound is 6 shillings and 8 pence: now 6 shillings and 8 pence are twice contained in 18 shillings, and there are 4 shillings and 8 pence over; which 4 shillings and 8 pence will be found by the 5th art. to be just $\frac{7}{10}$ of 6 shillings and 8 pence.

3dly, The whole number 10 divided by $2\frac{2}{3}$, that is $\frac{10}{1}$ divided by $\frac{8}{3}$, or multiplied into $\frac{3}{8}$, makes $\frac{30}{8}$, or $3\frac{3}{4}$.

4thly, $2\frac{2}{3}$ divided by $\frac{10}{1}$, or $\frac{8}{3}$ divided by $\frac{10}{1}$, or multiplied into $\frac{1}{10}$ makes $\frac{8}{30}$, or $\frac{4}{15}$.

5thly, $16\frac{1}{3}$ divided by $1\frac{1}{6}$, that is, $\frac{49}{3}$ divided by $\frac{7}{6}$ or multiplied into $\frac{6}{7}$ makes $\frac{294}{21}$, or 14.

Further observations concerning multiplication and division in fractions.

14. When two fractions are multiplied together, or one is divided by the other, it often happens, that though the original fractions be both in their least terms, yet the product, or quotient from them, shall be otherwise, and require a further reduction: as for instance, the fractions $\frac{5}{6}$ and $\frac{9}{10}$ are both in their least terms; and yet if they be multiplied together, their product $\frac{45}{60}$ is so far from being in it's least terms, that it may be reduced to $\frac{3}{4}$: so again in division, $\frac{9}{10}$ and $\frac{15}{16}$ are fractions both

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in their leaſt terms; and yet if the latter be divided by the former, the quotient $\frac{150}{144}$ is reducible to $\frac{25}{24}$. It may not be amiſs therefore, to enquire into the cauſe of this, and ſee whether the original fractions may not be ſo prepared beforehand, as that the product, or quotient, ſhall always come out in it's leaſt terms. Firſt then, as to the multiplication of $\frac{5}{6}$ and $\frac{9}{10}$; here it is eaſy to ſee, that the product of $\frac{5}{6}$ and $\frac{9}{10}$ multiplied together, will juſt amount to the ſame, as that of $\frac{5}{10}$ into $\frac{9}{6}$, the denominators of the fractions being interchanged; this I ſay is certain from the operation itſelf; for the ſame numbers are multiplied together in both caſes; but theſe laſt fractions are far from being in their leaſt terms, the former, $\frac{5}{10}$ being reducible to $\frac{1}{2}$, and the latter $\frac{9}{6}$ to $\frac{3}{2}$; but after theſe new fractions $\frac{5}{10}$ and $\frac{9}{6}$ are reduced to their leaſt terms $\frac{1}{2}$ and $\frac{3}{2}$; their product $\frac{3}{4}$ will be the ſame in value with that of the original fractions, and at the ſame time will be in it's leaſt terms. Thus then we ſee, that to have the product in it's leaſt terms, care muſt be taken, not only to reduce the original fractions as low as poſſible, but after that, to interchange their denominators, and then again to reduce theſe new fractions to their leaſt terms, and laſtly, to multiply theſe reduced fractions one into another.

The ſame manner of practice will alſo ſerve for diviſion, after it is reduced to the rule of multiplication: as for example; the quotient of $\frac{15}{16}$ divided by $\frac{9}{10}$, is the ſame with the product of $\frac{15}{16}$ multiplied into $\frac{10}{9}$; and this again, is the ſame with the product of $\frac{15}{9}$ multiplied into $\frac{10}{16}$, as above; but becauſe the fractions $\frac{15}{9}$ and $\frac{10}{16}$, are not in their loweſt terms, they muſt be reduced to $\frac{5}{3}$ and $\frac{5}{8}$ before it can be expected that their product $\frac{25}{24}$ ſhould be in it's leaſt terms. Thus we have reduced the two compendiums of multiplication and diviſion, not only to one rule inſtead of two, as they are commonly given out, but alſo to ſuch a rule as carries it's own evidence along with it.

N. B. What was here done by interchanging the denominators, and keeping the numerators in their places, may as well be done by interchanging the numerators, and keeping the denominators in their places, the reaſon of both being the ſame.

Of the rule of proportion in fractions.

15. The rule of proportion in fractions, is ſo much the ſame with the rule of proportion in whole numbers, that nothing more needs to be ſaid of it, except to illuſtrate it by an example or two.

Examples

Examples of the rule of proportion in fractions.

1st, If $\frac{1}{4}$ give $\frac{1}{5}$, what will $\frac{1}{6}$ give? Here $\frac{1}{5}$ and $\frac{1}{6}$ multiplied together give $\frac{1}{30}$; and this divided by $\frac{1}{4}$, (or multiplied by $\frac{4}{1}$) quotes $\frac{4}{30}$, or $\frac{2}{15}$, which is an anſwer to the queſtion.

2dly, If 2 $\frac{2}{3}$ give 3 $\frac{3}{4}$, what will 4 $\frac{4}{5}$ give? Theſe mixt numbers being by the 3d art. reduced to improper fractions, will ſtand thus: *If $\frac{8}{3}$ give $\frac{15}{4}$, what will $\frac{24}{5}$ give?* Here $\frac{15}{4}$ and $\frac{24}{5}$ multiplied together give $\frac{360}{20}$ or 18; and this divided by $\frac{8}{3}$, quotes 6 $\frac{3}{4}$, which is an anſwer to the queſtion.

3dly, If $\frac{1}{2}$ of a yard coſt $\frac{1}{3}$ of a pound, what will $\frac{1}{4}$ of an ell coſt? Here it muſt be obſerved, that an ell is $\frac{5}{4}$ of a yard, and conſequently that $\frac{1}{4}$ of an ell is $\frac{1}{4}$ of $\frac{5}{4}$, or $\frac{5}{16}$ of a yard; ſo that the queſtion may be ſtated thus: *If $\frac{1}{2}$ of a yard coſt $\frac{1}{3}$ of a pound, what will $\frac{5}{16}$ of a yard coſt?* Here $\frac{1}{3}$ and $\frac{5}{16}$ multiplied together make $\frac{5}{48}$, and this divided by $\frac{1}{2}$ quotes $\frac{5}{24}$ of a pound, or 4 ſhillings and twopence; which therefore is an anſwer to the queſtion.

The reduction of proportion from fractional to integral terms.

Whenever two fractions are propoſed, as $\frac{2}{3}$ and $\frac{4}{5}$, whoſe proportion is deſired in whole numbers, reduce the fractions firſt to the ſame denomination by the 8th art. that is, in the preſent caſe, to $\frac{10}{15}$ and $\frac{12}{15}$; then you will have $\frac{2}{3}$ to $\frac{4}{5}$ as $\frac{10}{15}$ is to $\frac{12}{15}$; but $\frac{10}{15}$ is to $\frac{12}{15}$ as 10 to 12, or as 5 to 6; therefore $\frac{2}{3}$ is to $\frac{4}{5}$ as 5 to 6: here we may obſerve, that though the finding of the common denominator be neceſſary for underſtanding t[illegible]ion of the rule, yet it is not at all neceſſary for the practice of it; [illegible]what purpoſe is it to find the common denominator, to throw it away again when we have done? In practice therefore, multiply the numerator of the fraction which is the firſt in the proportion, by the denominator of the ſecond, and then the numerator of the ſecond fraction, by the denominator of the firſt, and the two products will exhibit reſpectively, the proportion of the firſt fraction to the ſecond in whole numbers, as was evident in the foregoing example.

Of the extraction of roots in fractions.

16. As every fraction is ſquared, or multiplied into itſelf, by ſquaring both the numerator and denominator, (ſee art. 11.) ſo *e converſo* the ſquare

ſquare root of every fraction will be obtained by extracting the ſquare root both of the numerator and denominator: thus the ſquare of $\frac{3}{4}$ is $\frac{9}{16}$, and the ſquare root of $\frac{9}{16}$ is $\frac{3}{4}$. But here care muſt be taken, whenever the ſquare root of a fraction is to be extracted, that the fraction itſelf be firſt reduced to it's ſimpleſt terms, by the 7th art. otherwiſe the fraction may admit of a ſquare root, and yet this root may not be diſcovered: thus if it was required to extract the ſquare root of the fraction $\frac{18}{32}$, it would be impoſſible to obtain the root either of 18 or 32; and yet when this fraction is reduced to it's leaſt terms $\frac{9}{16}$, it's ſquare root will be found to be $\frac{3}{4}$.

When the ſquare root of a number cannot be extracted exactly, it is uſual to make an approximation by the help of decimals, or otherwiſe, and ſo to approach as near to the value of the true root as occaſion requires. Now in the caſe of a fraction, if the ſquare root of neither the numerator nor denominator can be exactly obtained, there will be no neceſſity however, for two approximations, becauſe ſuch a fraction may be eaſily reduced to another of the ſame value, whoſe denominator is a known ſquare: as for inſtance; ſuppoſe the ſquare root of $46\frac{1}{5}$, or $\frac{231}{5}$ was required: I multiply both the numerator and denominator of this fraction by 5, and ſo reduce it to $\frac{1155}{25}$: Here the denominator 25 is a known ſquare number, whoſe root is 5; and the ſquare root 1155 is 34 nearly; therefore, the ſquare root of the fraction propoſed is nearly $\frac{34}{5}$, or $6\frac{4}{5}$. But after all, the beſt way of extracting the ſquare root of a vulgar fraction, is by throwing it into a decimal fraction, as will be ſhewn hereafter.

Note. That whatever has here been ſaid concerning the extraction of the ſquare root in fractions, may eaſily be applied, *mutatis mutandis*, to the extraction of the cube root, &c.

Of decimal fractions.

And firſt of their notation.

17. A decimal fraction is a fraction whoſe denominator is 10, or 100, or 1000, or 10000, &c. and this denominator is never expreſſed, but always underſtood by the place of the figure it belongs to: for as all figures on the left hand of the place of units, riſe in their value, according to their diſtances from it, in a decuple proportion; ſo all figures on the right hand of the place of units, ſink in their value in a ſubdecuple proportion: as for inſtance; the number 345.6789, where 5 ſtands

in

in the place of units, is to be read thus; *three hundred forty five, six tenths, seven hundredth parts, eight thousandth parts, nine tenthousandth parts:* or the decimal parts may be read thus; *six thousand seven hundred eighty nine tenthousandth parts*; the denominator being ten thousand, because the last figure 9, according to the former way of reckoning, stands in the place of tenthousandth parts. The reason of this latter way of reading is plain; for $\frac{6}{10}$ are $\frac{6000}{10000}$, and $\frac{7}{100}$ are $\frac{700}{10000}$, and $\frac{8}{1000}$ are $\frac{80}{10000}$, and $\frac{6000}{10000}$, $\frac{700}{10000}$, $\frac{80}{10000}$ and $\frac{9}{10000}$, all added together, make $\frac{6789}{10000}$.

Cyphers are used in the expression of decimals as well as whole numbers, and for the same reason. Thus .067 may be read either *no tenths, six hundredth parts, seven thousandth parts*; or *sixty seven thousandth parts*. But cyphers on the right hand of a decimal number (if nothing follows them) are as insignificant as cyphers on the left hand of a whole number; and yet cyphers are sometimes placed after decimals, for the sake of regularity, or when we want to increase the number of decimal places.

From what has here been said, it will be easy to multiply or divide any number by 10, 100, 1000, *&c* only by removing the separating point towards the right or left hand. Thus the number 345.6789 being multiplied by 10, becomes 3456.789; and being multiplied by 100, becomes 34567.89: and the same number 345.6789 being divided by 10, becomes 34.56789; and being divided by 100, becomes 3.456789: thus again, the number 345 being divided by 10000, becomes .0345; for to divide by 10000, is the same thing as to remove the separating point 4 degrees towards the left hand, if there be any separating point in the number given; but if there be none, as in the present case, then to put a separating point four degrees towards the left hand, which in this example cannot be done, but by the help of a cypher in the first decimal place.

Of the addition and subtraction of decimal fractions.

18. The chief advantage of decimal arithmetic above that of common fractions, consists in this, that in decimals, all operations are performed as in whole numbers: this will presently appear from the several parts of decimal arithmetic, as they come now to be treated of in order; and first of addition and subtraction.

Addition and subtraction in decimals are performed after the same manner as in whole numbers, care being taken, that like parts be placed under

der one another: as for example, .567 are added to .89 thus;

.89	subtracted thus;	.89	or thus;	.890
.567		.567		.567
1.457:		.323;		.323.

Of the multiplication of decimal fractions.

19. Multiplication in decimals is also performed as in whole numbers, no regard being had to the decimals as such, till the product is obtained; but then, so many decimal places must be cut off from the right hand of the product, as are contained both in the multiplicator and multiplicand: as for instance; let it be required to multiply 4.56 by 2.3: here considering both factors as whole numbers, I multiply 456 by 23, and find the product to be 10488; but then considering that there was one decimal in the multiplicator, and two in the multiplicand, I cut off three decimal places from the right hand of the product, and the true product stands thus; 10.488.

To shew the reason of this operation, let the two factors be reduced to simple fractions according to the common way, and we shall have 2.3 equal to $\frac{23}{10}$, and 4.56 equal to $\frac{456}{100}$, and these two fractions multiplied together make $\frac{10488}{1000}$; divide by 1000, which is done by cutting off the three last figures, according to art. the 17th, and the quotient will be 10.488. Another example may be this: let it be required to multiply 45600 by .23: the product of 45600 multiplied by 23 is 1048800: but as there were two decimals in the given multiplicator, and none in the multiplicand; I cut off two decimal places from the last product, and the true product will be found to be 10488.00, or 10488. Lastly, let it be required to multiply .000456 by .23: here neglecting the initial cyphers in the multiplicand, I multiply 456 by 23, and the product is 10488: then I consider, that there were two decimal places in the multiplicator, and six in the multiplicand, and consequently that eight decimal places are to be cut off from the last product: but the last product consists of only 5 places; therefore I place three cyphers to the left hand, with the separating point before them, and so make the true product .00010488.

There are various compendiums of this sort of multiplication to be met with in *Oughtred* and others; but they are such, as by a little exercise, any one tolerably well grounded in this part of Arithmetic will easily discover of himself as they lie in his way.

Of

Of the diviſion of decimal fractions.

20. Diviſion in decimal fractions is performed, firſt by conſidering them as whole numbers, and dividing accordingly; and then cutting off from the right hand of the quotient, as many decimal places as the dividend hath more than the diviſor. The reaſon whereof is manifeſt from the 19th article: for ſince the diviſor and quotient multiplied together are to make the dividend, the diviſor and quotient ought to have as many decimal places between them, as there are in the dividend; therefore the quotient alone ought to have as many decimal places as the dividend hath more than the diviſor.

Example the 1*ſt*; Let it be propoſed to divide 10.488 by 2.3: here dividing the whole number 10488 by the whole number 23, I find the quotient to be 456: but then conſidering that there were 3 decimal places in the dividend, and but one in the diviſor, I cut off two places from the right hand of the quotient, and ſo make the true quotient 4.56.

Example 2*d*; Let it be propoſed to divide 5678.9 by .06: here becauſe there are two decimal places in the diviſor, and but one in the dividend, I ſupply the deficient place by putting a cypher after the dividend, thus, 5678.90; then dividing the whole number 567890, by the whole number 6, (for ſince 6 is now conſidered as a whole number, the cypher before it may be neglected;) I find the quotient to be 94648, which is not to be ſunk, becauſe the dividend was made to have as many decimal places as the diviſor; but as this quotient is not exact, if for a greater degree of exactneſs I would continue it to any number of decimal places, ſuppoſe 2, inſtead of one cypher after the diviſor, I would have put three, and then the quotient would have come out 94648.33, and this quotient is much more exact than the former, as lying between 94648.33 and 94648.34: but it ought further to be obſerved concerning this quotient, that if the diviſion was to be continued *in infinitum*, the figures in the decimal places would be all 3's: this is evident from the work; for the two laſt dividuals are the ſame, and therefore they muſt all be the ſame.

To reduce a vulgar fraction to a decimal fraction.

21. Since every fraction may be conſidered as the quotient of the numerator divided by the denominator, (ſee art. 13th,) we have an eaſy rule for reducing a vulgar fraction to a decimal fraction, which is as follows: put as many cyphers after the numerator, as are equal in

number, to the number of decimal places whereof you intend your reduced fraction to consist, and call these cyphers decimal; and then dividing the numerator by the denominator, the quotient will be a decimal number equal to the fraction first proposed, or perhaps a mixt number, if the fraction proposed was an improper one.

Example 1*st*; Let this fraction $\frac{3}{49}$ be proposed to be reduced to a decimal one consisting of four decimal places: here putting 4 decimal cyphers after the numerator 3, I divide 3 .0000 by 49, and the quotient uncorrected is 612: but now considering that there were 4 decimal places in the dividend, and none in the divisor, and consequently that four decimal places are to be cut off from the quotient, whereas it consists but of three; I supply this defect of places by a cypher at the left hand, and so make the quotient .0612.

Example 2*d*; Let this fraction $\frac{7}{16}$ be proposed to be reduced to a decimal fraction consisting, if possible, of six places: here dividing 7 .000000 by 16, I find the true quotient to be .4375 the two last cyphers in the dividend being useless.

Note. When this division runs *ad infinitum*, it will be impossible for the reduction to be exact in a finite number of terms; but an approximation may be made that shall come nearer to the quotient than the least assignable difference, by taking more and more terms.

To reduce the decimal parts of any integer to such other parts as that integer is usually divided into.

22. To explain this rule, and to give an example of it at the same time; let .345 of a pound sterling, that is, three hundred forty five thousandth parts of a pound be given to be reduced into shillings, pence and farthings: here then I observe, that as any number of pounds multiplied by 20 will give as many shillings as are equal to the pounds, so any decimal parts of a pound multiplyed by 20, will give as many shillings, and decimal parts of a shilling, as are equivalent to the decimal parts of a pound; and so on as to pence and farthings: multiplying therefore .345 by 20, the product is 6 and .900, or 6.9, which signifies, that .345 of a pound are equivalent to six shillings and nine tenths of a shilling, which is usually written thus; 6 .9 shillings: again, multiplying this last decimal .9 by 12 for pence, I find that .9 of a shilling are equivalent to 10 .8 pence: lastly, multiplying .8 by 4 for farthings, I find that .8 of a penny are equivalent to 3 .2 farthings; as for the .2 of a farthing, I neglect it, there being no lower denomination,

or

or at leaſt, not intending to deſcend any lower; and ſo I find .345 of a pound to amount to ſix ſhillings and tenpence three farthings.

To reduce the common parts of any integer into equivalent decimal parts of the ſame.

23. This reduction being the reverſe of the former, it might be performed by diviſion, as that was by multiplication; but when all things are conſidered, I do not know whether the following method may not be thought as eaſy, and as intelligible an any: let it then be required to reduce 2 hours, 34 minutes, 56 ſeconds, into equivalent decimal parts of a day. Now in one day there are 86400 ſeconds; and in two hours, 34 minutes, 56 ſeconds, there are 9296 ſeconds; therefore two hours, 34 minutes, 56 ſeconds, are equivalent to $\frac{9296}{86400}$ of one day: reduce this vulgar fraction to an equivalent decimal, by the laſt article but one, and you will find it to be .10759; therefore 2 hours, 34 minutes, 56 ſeconds, are equivalent to .10759 of one day. But there is one article ſtill remains to be adjuſted, and that is, to how many decimal places the foregoing fraction muſt be reduced, ſo as to expreſs accurately enough, the parts of a day to a ſecond of time. Now to know this, I conſider that one ſecond of time is $\frac{1}{86400}$ of one day; therefore I reduce $\frac{1}{86400}$ to a decimal fraction, at leaſt as far as to the firſt ſignificant figure, and find it to be .00001; whence I conclude, that to expreſs the parts of a day to a ſecond of time by any decimal, that demal muſt not conſiſt of fewer than 5 places, becauſe there were 5 places in the decimal fraction .00001. Now to ſhew that the decimal fraction above found, to wit, .10759 expreſſes the time propoſed to a ſecond, reduce it back again, by the laſt art. and you will find it amount to 2 hours, 34 minutes, 55.8 ſeconds.

For another example, let us take the reverſe of that in the laſt art. that is, let it be propoſed to reduce ſix ſhillings ten pence 3.2 farthings, into equivalent decimal parts of a pound: one pound contains 960 farthings, or 9600 tenths of a farthing; and 6 ſhillings, 10 pence 3.2 farthings, contain 3312 tenths of a farthing; therefore 6 ſhillings, 10 pence 3.2 farthings, are eqvivalent to $\frac{3312}{9600}$ of a pound; but $\frac{1}{9600}$ being reduced to a decimal, is .0001 &c. wherein the firſt ſignificant figure is in the 4th place; therefore I reduce the fraction $\frac{3312}{9600}$, to four decimal places, and they amount to .3450, that is, .345 of a pound; ſo that in

in this particular caſe, three decimal places are ſufficient to expreſs exactly the ſum propoſed.

Of the extraction of the ſquare root in decimal fractions.

24. Having treated of the multiplication and diviſion of decimal fractions, it would be altogether needleſs to ſay any thing concerning the rule of proportion, which is but a particular application of both: therefore I ſhall now paſs on to the extraction of the ſquare root, at leaſt ſo far as it concerns decimal fractions. There are but few ſquare numbers, or ſuch as will admit of an exact ſquare root, in compariſon of the reſt; and therefore, whenever a number is propoſed to have it's ſquare root extracted, the artiſt muſt firſt determine with himſelf, to how many decimal places it is proper the root ſhould be continued; and then by annexing decimal cyphers, if need be, to the right hand of the number propoſed, he muſt make twice as many decimal places there, as the root is to conſiſt of; after this, he muſt put a point over the place of units, and then paſſing by every other figure, he muſt point in like manner all the reſt, both to the right hand, and to the left: by this means, the number will be prepared, and the ſquare root may be extracted as in whole numbers, provided that ſo many decimal places be cut off from the root when obtained, as were firſt deſigned.

Example *1ſt*; Let the root of 2345.6 be required to two decimal places. The number when prepared, ſtands thus, 2345.6000, or as a whole number, thus, 23456000; and it's ſquare root, when extracted, will be 4843 nearly; and therefore 48.43 will be the root ſought. To try this root 48.43, multiply it into itſelf, and the 4 firſt figures of the ſquare will be 2345, which are all true, nor can it be expected any more ſhould be ſo, becauſe there were but four places true in the root, no notice being taken of the reſt: but had the root been extracted true to 5 places, that is, to as many places as the original ſquare conſiſted of, it would then have been 48.431; multiply this number into itſelf, and 5 of the firſt figures of the product, taken with the leaſt error, will be 2345.6, which is the original ſquare itſelf.

Example *2d*; Let the root of .0023456 be required to 5 decimal places. Here putting a cypher in the place of units to direct the punctation, thus, 0.0023456000, I extract the ſquare root of 23456000 as of a whole number, and find it to be 4843, as above: but conſidering

that

that this root is to be ſunk 5 places, I put a cypher to the left hand, and ſo make the true root .04843.

That the ſuppoſed ſquare ought to have twice as many decimal places as the root, is evident, both *à priori*, and *à poſteriori* : *à priori*, becauſe in extracting the ſquare root, two figures are brought down from the ſquare for every ſingle figure gained in the root; and *à poſteriori*, becauſe the root multiplied into itſelf is to produce the ſquare; and therefore, from the nature of multiplication, the ſquare ought to have twice as many decimal places as the root.



THE ELEMENTS of ALGEBRA

BOOK I.

The Definition of Algebra.

Article 1. I SHALL not here detain the young ſtudent with a long hiſtorical account of the riſe and progreſs of Algebra; nor even ſo much as with either the etymology or ſignification of the word, which would contribute but very little to his information, till he has made a further progreſs in the ſcience itſelf, and whereof he will find enough in Dr. *Wallis* and others. Nor indeed is it a ſubject altogether ſo proper at this time to be inſiſted upon; this art, like many others, having now conſiderably outgrown it's name, and being often employed in arithmetical operations very different from what it's name imports. All I ſhall advance then, by way of definition is, that *Algebra*, in the modern ſenſe of the word, *is the art of computing by ſymbols*, that is, generally ſpeaking, by letters of the alphabet; which for the ſimplicity and diſtinctneſs both of their ſounds and characters, are much more commodious for this purpoſe than any other ſymbols or marks whatever.

In this way of notation, it is uſual to ſubſtitute letters not only for ſuch quantities as are unknown, and conſequently, ſuch as cannot well be repreſented otherwiſe, but alſo for known quantities themſelves, in order to keep them diſtinct one from another, and to form general concluſions. As for inſtance; ſuppoſe it was demanded of me, what two numbers are thoſe, whoſe ſum is 48, and whoſe difference is 14: here, if I only put x, or ſome other letter for one of the unknown quantities, and uſe the known ones 48 and 14 as I find them in the problem, I ſhall

only come to this particular conclufion, to wit, that the greater number is 31, and the lefs 17, which numbers will anfwer both the conditions of the problem. But if inftead of the known numbers 48 and 14, I fubftitute the general quantities *a* and *b* refpectively, and fo propofe the problem thus; *What two numbers are thofe, whofe fum is* a, *and whofe difference is* b? I fhall then come to this general conclufion, *viz.* that *Half the fum of* a *and* b *will be the greater number, and half their difference will be the lefs:* which general theorem will fuit not only the particular cafe abovementioned, but alfo all other cafes of this problem that can poffibly be propofed. How I come by thefe two conclufions, will be fufficiently fhewn in the courfe of this work; as alfo many other advantages attending this way of fubftituting letters for known quantities, befides thofe already mentioned.

What I have here faid, was only to illuftrate in fome meafure, the definition already given of Algebra, and to fhew, that letters are there ufed, not fo much to fignify particular quantities as fuch, as to fignify the relation they have to one another in any problem or computation. From all which it may be obferved, that letters reprefent quantities in Algebra juft in the fame manner as they do perfons in common life, when two or more perfons are diftinctly to be confidered with regard to any compact, law-fuit, or in any other relation whatever.

N. B. A fingle quantity is fometimes reprefented by two or more letters, when it is confidered as the product of the quantities fignified by thofe letters fingly: thus ab is the product of the multiplication of a and b; and abc is the product arifing from the continual multiplication of a, b and c. But of this more particularly under the head of multiplication.

Of affirmative and negative quantities in Algebra.

2. Algebraic quantities are of two forts, affirmative and negative: an affirmative quantity is a quantity greater than nothing, and is known by this fign $+$; a negative quantity is a quantity lefs than nothing, and is known by this fign $-$: thus $+a$ fignifies that the quantity a is affirmative, and is to be read thus, *plus a*, or more *a*: $-b$ fignifies that the quantity b is negative, and muft be read thus, *minus b*, or lefs *b*.

The poffibility of any quantitie's being lefs than nothing is to fome a very great paradox, if not a downright abfurdity; and truly fo it would be, if we fhould fuppofe it poffible for a body or fubftance to be lefs than nothing. But quantities, whereby the different degrees of qualities are eftimated, may be eafily conceived to pafs from affirmation through nothing into negation. Thus a perfon in his fortunes may be faid to be worth 2000 pounds, or 1000, or nothing, or -1000, or -2000; in

in which two laſt caſes he is ſaid to be 1000 or 2000 pounds worſe than nothing: thus a body may be ſaid to have 2 degrees of heat, or one degree, or no degree, or — one degree, or — two degrees: thus a body may be ſaid to have two degrees of motion downwards, or one degree, or no degree, or — one degree, or — two degrees, *&c.* Certain it is, that all contrary quantities do neceſſarily admit of an intermediate ſtate, which alike partakes of both extremes, and is beſt repreſented by a cypher or 0: and if it is proper to ſay, that the degrees on either ſide this common limit are greater than nothing; I do not ſee why it ſhould not be as proper to ſay of the other ſide, that the degrees are leſs than nothing; at leaſt in compariſon to the former. That which moſt perplexes narrow minds in this way of thinking, is, that in common life, moſt quantities loſe their names when they ceaſe to be affirmative, and acquire new ones ſo ſoon as they begin to be negative: thus we call negative goods, debts; negative gain, loſs; negative heat, cold; negative deſcent, aſcent, *&c*: and in this ſenſe indeed, it may not be ſo eaſy to conceive, how a quantity can be leſs than nothing, that is, how a quantity under any particular denomination, can be ſaid to be leſs than nothing, ſo long as it retains that denomination. But the queſtion is, whether, of two contrary quantities under two different names, one quantity under one name may not be ſaid to be leſs than nothing, when compared with the other quantity, though under a different name; whether any degree of cold may not be ſaid to be further from any degree of heat, than is lukewarmth, or no heat at all. Difficulties that ariſe from the impoſition of ſcanty and limited names, upon quantities which in themſelves are actually unlimited, ought to be charged upon thoſe names, and not upon the things themſelves, as I have formerly obſerved upon another occaſion; ſee introduction, art. 11. In Algebra, where quantities are abſtractedly conſidered, without any regard to degrees of magnitude, the names of quantities are as extenſive as the quantities themſelves; ſo that all quantities that differ only in degree one from another, how contrary ſoever they may be one to another, paſs under the ſame name; and affirmative and negative quantities are only diſtinguiſhed by their ſigns, as was obſerved before, and not by their names; the ſame letter repreſenting both: theſe ſigns therefore in Algebra carry the ſame diſtinction along with them as do particles and adjectives ſometimes in common language, as in the words convenient and inconvenient, happy and unhappy, good health and bad health, *&c.*

Theſe affirmative and negative quantities, as they are contrary to one another in their own natures, ſo likewiſe are they in their effects, a conſideration which if duly attended to, would remove all difficulties concerning the ſigns of quantities ariſing from addition, ſubtraction, multi-

plication, division, &c: for the result of working by affirmative quantities in all these operations is known; and therefore like operations in negative quantities may be known by the rule of contraries.

Before we proceed any further, it may not be amiss to advertise, that if a quantity has no sign before it, it must always be taken to be affirmative; and that if it has no numeral coefficient before it, unity must always be understood: thus $2a$ signifies $+2a$, and a signifies $1a$ or $+1a$.

By the numeral coefficient of a quantity, I mean the number or fraction by which that quantity is multiplied: thus $2a$ signifies twice a, or a taken twice, and the coefficient is 2: $\frac{3}{4}a$, or $\frac{3a}{4}$ signifies $\frac{3}{4}$ of the quantity a, and the coefficient is $\frac{3}{4}$.

N. B. The sign of a negative quantity is never omitted, nor the sign of an affirmative one, except when such an affirmative quantity is considered by itself, or happens to be the first in a series of quantities succeeding one another: thus we do not often mention the quantity $+a$, but the quantity a; nor the series $+a-b-c+d$, but the series $a-b-c+d$. We shall now consider the several operations of algebraic quantities.

Of the addition of algebraic quantities.

3. This article I shall divide into several paragraphs: as

1st, Whenever two or more quantities of the same denomination, and which have the same sign before them, are to be added together, put down the sum of their numeral coefficients with the common sign before it, and the common denominator after it: thus $+2a$ and $+3a$ added together make $+5a$, for the same reason as 2 dozen and 3 dozen added together make 5 dozen: thus again, $-3ab$, $-4ab$ and $-5ab$ when added together make $-12ab$; for the same reason as several debts added together make a greater debt.

2d, If two quantities of the same denomination which have different signs before them are to be added together, put down only the difference of their numeral coefficients with the common denominator after it, and the sign of the greater quantity before it: for in this case, the quantities to be added being contrary one to another, the less quantity, on which side soever it lies, will always destroy so much of the other, as is equal to itself. Thus $+5a$ added to $-2a$ makes $+3a$; as if a person owes me 5000 pounds upon one account, to whom I owe 2000 upon another, the balance upon the whole will be 3000 pounds on my side. If it be objected, that this is subtraction, and not addition; I answer, that the addition of $-2a$ will at any time have the same effect as the subtraction

ſubtraction of $+2a$: but I deny that the addition of $-2a$ is the ſame, or will have the ſame effect as the ſubtraction of $-2a$. Other examples of this caſe may be theſe; $+7a$ added to $-7a$ gives 0; $+3a$ added $-12a$ gives $-9a$; $+a$ added to $-5a$ gives $-4a$; $+5a$ added to $-a$ gives $+4a$; $+\frac{1}{3}a$ added to $-\frac{1}{4}a$ gives $+\frac{1}{12}a$, &c.

3*d*, When many quantities of the ſame denomination are to be added together, whereof ſome are affirmative and ſome negative, reduce them firſt to two, by adding all the affirmative quantities together, and all the negative ones, and then to one by the laſt paragraph. Thus $+10a-9a+8a-7a$ when added together make $2a$; for $+10a$ and $+8a$ make $+18a$, and $-9a$ and $-7a$ make $-16a$; and $+18a$ and $-16a$ make $+2a$.

4*th*, Quantities of different denominations will not incorporate, and therefore cannot otherwiſe be added together, than by placing them in any order one after another with their proper ſigns before them, except the firſt, whoſe ſign, if affirmative, may be omitted. Thus $+2a$ and $-3b$ and $+4c$ and $-5d$, when added together, make $2a-3b+4c-5d$: thus a and b added together make $a+b$; and hence it is, that whenever two quantities are found with this ſign $+$ betwixt them, it ſignifies the ſum ariſing from the addition of thoſe two quantities together: thus if a ſtands for 7, and b ſtands for 3, $a+b$ will ſtand for 10, and ſo of the reſt: but if $-b$ is to be added to a, the ſum muſt be written down thus, $a-b$; for to add $-b$, is the ſame as to ſubtract $+b$.

5*th*, Compound quantities, whoſe members are all of different denominations, are likewiſe incapable of being added any other way, than by being placed one after another without altering their ſigns: thus $3a+4b$ added to $5c-6d$ can only make $3a+4b+5c-6d$. But if the members are not all of different denominations, it may then be convenient to place one compound quantity under another, with like parts under like, as far as it can be done, as in the following examples:

$$\begin{array}{r} a+b \\ a-b \\ \hline 2a\quad *.\ddagger \end{array}$$

‡ For a and a added together make $2a$; and $+b$ and $-b$ added together deſtroy one another, and ſo make 0 or $*$; which character in Algebra is always uſed to ſignifie a vacant place.

$$\begin{array}{l} 2x-3a+4b\ -5c+6d-7e\quad * \\ 10x+9a-8b\ -7c-6d\quad *-5f \\ \hline 12x+6a-4b-12c\quad *-7e-5f. \end{array}$$

Note, That in the addition, ſubtraction and multiplication of compound algebraic quantities, it matters little which way the work is carri-
ed

ed on, whether from right to left, or from left to right, because here are no reserves made for higher places.

Of the subtraction of algebraic quantities.

4. Whenever a simple algebraic quantity is to be subtracted from another quantity, whether simple or compound, first change the sign of the quantity to be subtracted, that is, if it it be affirmative, make it, or at least call it negative, and *vice versa*, and then add it so changed to the other: for since (as was before hinted) the subtracting of any one quantity from another, is the same in effect as adding the contrary, and since changing the sign of the quantity to be subtracted, renders that quantity just contrary to what it was before, it is evident, that after such a change it may be added to the other, and that the result of this addition will be the same with that of the intended subtraction. Thus may the rule of subtraction, by changing the sign of the quantity to be subtracted, be at any time changed into that of addition, just as the rule of division in fractions by inverting the terms of the divisor, was changed into that of multiplication. As for example, $+b$ subtracted from a leaves $a-b$, because $-b$ added to a makes $a-b$; so that $a-b$ may be considered either as the sum of a and $-b$ added together, or as the remainder of $+b$ subtracted from a, or as the difference between a and b, or as the excess of a above b, all which amount to the same thing: as if a signifies 7, and b 3, $a-b$ must stand for 4, and so of the rest.

The rule of subtraction here given is universal, though there will not be always occasion to have recourse to it: for suppose $3a$ is to be subtracted from $7a$, every ones common sense will inform him, that here must remain $4a$, just as threescore subtracted from sevenscore leaves fourscore.

Other examples of algebraic subtraction may be these that follow.

1st, $7a$ subtracted from $5a$ leaves $-2a$, because $-7a$ added to $+5a$ makes $-2a$, by the 2d paragraph of the last article.

2d, $9a$ subtracted from 0 leaves $-9a$, because $-9a$ added to 0 makes $-9a$.

3d, $12a$ subtracted from $-3a$ leaves $-15a$, because $-12a$ added to $-3a$ makes $-15a$, by the first paragraph of the last article.

4th, $-3a$ subtracted from $-8a$ leaves $-5a$, because $+3a$ added $-8a$ makes $-5a$.

5th,

5th, $-7a$ subtracted from $-3a$ leaves $+4a$, because $+7a$ added to $-3a$ makes $+4a$.

6th, $-6a$ subtracted from 0 leaves $+6a$, because $+6a$ added to 0 makes $+6a$.

7th, $-5a$ subtracted from $+5a$ leaves $+10a$, because $+5a$ added to $+5a$ makes $+10a$.

8th, $-b$ subtracted from a leaves $a+b$, because $+b$ added to a makes $a+b$, by the 4th paragraph of the last article.

9th, -2 subtracted from 7 leaves 9, because $+2$ added to 7 makes 9.

From the first of these examples it appears, that a greater quantity may be taken out of a less, but then the remainder will be negative; just as a gamester that has but 5 guineas about him may loose 7, but then there will remain a debt of 2 guineas upon him. By the last example it appears, that -2 subtracted from 7 leaves 9, that is, that if a negative quantity be subtracted from an affirmative one, the affirmative quantity will be so far from being diminished thereby, that it will be increased; a principle which I fear will be found somewhat hard of digestion, especially by weak constitutions: therefore to strengthen my patient as far as lies in my power, I shall suggest to him the following considerations:

1st, In any subtraction, if the remainder and the less number added together, make the greater, the subtraction is just: but in our case, the remainder 9 added to the less number -2 makes the greater number 7; therefore -2 subtracted from 7 leaves 9.

2dly, In all subtraction whatever, the remainder is the difference betwixt the greater number and the less: but the difference between $+7$ and -2 is 9; therefore -2 subtracted from $+7$ leaves 9.

3dly, 7 is equal to $\overline{9-2}$ by the second paragraph of the last article; therefore -2 subtracted from 7 will have the same remainder as -2 subtracted from $\overline{9-2}$: but -2 subtracted from $\overline{9-2}$ leaves 9; therefore -2 subtracted from 7 leaves 9. In short, the taking away a defect in any case whatever, will amount to the same, as adding something real: as if an estate be incumbered with a mortgage or a rent-charge upon it, whoever takes off the incumberance, just so much increases the value of the estate.

4thly, The less there is taken from 7, the more will be left: if nothing be taken, there will remain 7; therefore if less than nothing be taken, there ought to remain more than 7.

5thly, If after all that has been said, or perhaps all that can be said in this abstracted way, some scruples still remain, let us apply the principle we have already advanced, and try whether we shall meet with any better

ter ſucceſs that way. Let it then be required to ſubtract the compound quantity $a - 2$ from the compound quantity $6a + 7$: in order to this, I place a under $6a$, and -2 under 7, and then ſubtract as follows; a from $6a$ and there remains $5a$, -2 from 7 and (if our aſſertion be true) there remains 9; therefore the whole remainder is $5a + 9$. Now I dare appeal to every ones common ſenſe, whether this ſubtraction be not juſt: for certain it is, that if a be ſubtracted from $6a + 7$, the remainder will be $5a + 7$; and if ſo, then it is as certain, that if $a - 2$ be ſubtracted, which is leſs than the former by 2, the remainder will be greater by 2, that is, $5a + 9$. But to proceed:

Other examples of the ſubtraction of compound algebraic quantities may be theſe.

$$\begin{array}{r} a + b \\ a - b \\ \hline * + 2b\ddagger \end{array} \qquad\qquad \begin{array}{r} * + 12 \\ 3a + 7 \\ \hline -3a + 5. \end{array}$$

‡ Thus $7 - 3$, or 4, ſubtracted from $7 + 3$, or 10, leaves twice 3, or 6.

From	$12x + 6a - 4b - 12c \quad * - 7e - 5f$
Take	$2x - 3a + 4b - 5c + 6d - 7e \quad *$
Remains	$10x + 9a - 8b - 7c - 6d \quad * - 5f$
Proof	$12x + 6a - 4b - 12c \quad * - 7e - 5f.$

If never a member of the ſubtrahend be found to be of the ſame denomination with any member of the number from whence the ſubtraction is to be made, change the ſign of every member of the ſubtrahend, and then add it to the other. As if $5c - 6d$ is to be ſubtracted from $3a - 4b$, firſt change the ſign of $5c - 6d$, and make it $-5c + 6d$, and then add it to the other, and you will have $3a - 4b - 5c + 6d$ for a remainder.

Of the multiplication of algebraic quantities.

And firſt, how to find the ſign of the product in multiplication, from thoſe of the multiplicator and multiplicand given.

5. Before we can proceed to the multiplication of algebraic quantities, we are to take notice, that if the ſigns of the multiplicator and multiplicand be both alike, that is, both affirmative, or both negative, the product will be affirmative, otherwiſe it will be negative: thus $+4$ multiplied into $+3$, or -4 into -3 produces in either caſe $+12$: but

but —4 multiplied into +3, or +4 into —3 produces in either case —12.

If the reader expects a demonstration of this rule, he must first be advertised of two things: *first*, that numbers are said to be in arithmetical progression, when they increase or decrease with equal differences, as 0, 2, 4, 6; or 6, 4, 2, 0; also as 3, 0, —3; 4, 0, —4; 12, 0,—12; or —12, 0, +12: whence it follows, that three terms are the fewest that can form an arithmetical progression; and that of these, if the two first terms be known, the third will easily be had: thus if the two first terms be 4 and 2, the next will be 0; if the two first be 12 and 0, the next will be —12; if the two first be —12 and 0, the next will be +12, &c.

2dly, If a set of numbers in arithmetical progression, as 3, 2 and 1, be successively multiplied into one common multiplicator, as 4, or if a single number, as 4, be successively multiplied into a set of numbers in arithmetical progression, as 3, 2 and 1, the products 12, 8 and 4, in either case, will be in arithmetical progression.

This being allowed, (which is in a manner self-evident,) the rule to be demonstrated resolves itself into four cases:

1st, That +4 multiplied into +3 produces +12.

2dly, That —4 multiplied into +3 produces —12.

3dly, That +4 multiplied into —3 produces —12.

And *lastly*, that —4 multiplied into —3 produces +12. These cases are generally expressed in short thus: first + into + gives +; secondly — into + gives —; thirdly + into — gives —; fourthly — into — gives +.

Case 1st. That +4 multiplied into +3 produces +12, is self-evident, and needs no demonstration; or if it wanted one, it might receive it from the first paragraph of the 3d article; for to multiply +4 by +3 is the same thing as to add 4+4+4 into one sum; but 4+4+4 added into one sum give +12, therefore +4 multiplied into +3, gives +12.

Case 2d. And from the second paragraph of the 3d art. it might in like manner be demonstrated, that —4 multiplied into +3 produces —12: but I shall here demonstrate it another way, thus: multiply the terms of this arithmetical progression 4, 0, —4, into +3, and the products will be in arithmetical progression, as above; but the two first products are 12 and 0; therefore the third will be —12; therefore —4 multiplied into +3, produces —12.

Case 3d. To prove that +4 multiplied into —3 produces —12; multiply +4 into +3, 0, and —3 successively, and the products will be in arithmetical progression; but the two first are 12 and 0, there-

fore the third will be — 12; therefore + 4 multiplied into — 3 produces — 12.

Case 4th. Lastly, to demonstrate, that — 4 multiplied into — 3 produces + 12, multiply — 4 into 3, 0, and — 3 successively, and the products will be in arithmetical progression; but the two first products are — 12 and 0, by the second case; therefore the third product will be + 12; therefore — 4 multiplied into — 3 produces + 12.

Cas. 2d.				*Cas. 3d,*		
+ 4,	0,	— 4		+ 4,	+ 4,	+ 4
+ 3,	+ 3,	+ 3		+ 3,	0,	— 3
+ 12,	0,	— 12.		+ 12,	0,	— 12.

Cas. 4th,		
— 4,	— 4,	— 4
+ 3,	0,	— 3
— 12,	0,	+ 12.

These 4 cases may be also more briefly demonstrated thus: + 4 multiplied into + 3 produces + 12; therefore — 4 into + 3, or + 4 into — 3 ought to produce something contrary to + 12, that is, — 12: but if — 4 multiplied into + 3 produces — 12, then — 4 multiplied into — 3 ought to produce something contrary to — 12, that is, + 12; so that this last case, so very formidable to young beginners, appears at last to amount to no more than a common principle in Grammar, to wit, that two negatives make an affirmative; which is undoubtedly true in Grammar, though perhaps it may not always be observed in languages.

Of the multiplication of simple algebraic quantities.

6. These things premised, the multiplication of simple algebraic quantities is performed, first by multiplying the numeral coefficients together, and then putting down, after the product, all the letters in both factors, the sign (when occasion requires) being prefixed as above directed. Thus $4b$ multiplied into $3a$ produces $12ab$.

Though this kind of language (for it is no more) like all others, be purely arbitrary, yet that a more rational one could not have been invented for this purpose, will appear by the following consideration. If any quantity, as b, is to be multiplied by any number, as 2, 3, or 4, the product cannot be better represented than by $2b$, $3b$, $4b$, &c; therefore if b is to be multiplied by a, the product ought to be called ab: but if b multiplied into a produces ab, then $4b$ multiplied into a ought to produce 4 times as much, that is, $4ab$; lastly, if $4b$ multiplied into

a produces $4ab$, then $4b$ multiplied into $3a$ ought to produce 3 times as much, that is, $12ab$.

Hence it is, that whenever in Algebra two or more letters are found together, as they ſtand in a word without any thing between them, they ſignifie the product ariſing from a continual multiplication of the quantities repreſented by them: thus ab ſignifies the product of a and b multiplied together; and abc ſignifies the product of the quantity ab multiplied into c: thus aa ſignifies the product of a multiplied into itſelf, or the ſquare of a, and not $2a$; and therefore whoever ſhews himſelf unable to diſtinguiſh betwixt $2a$ and aa, diſcovers as great a weakneſs as one that is not able to diſtinguiſh betwixt 2 dozen and a dozen dozen or 12 times 12.

It is a matter of no great conſequence in what order the letters are placed in a product; for ab and ba differ no more from one another than 3 times 4, and 4 times 3: and yet it is convenient that a method be obſerved, leſt like quantities be ſometimes taken for unlike; therefore the beſt way will be, to give thoſe letters the precedency in a product, that have it in the alphabet; except when an unknown quantity is multiplied by ſome known one, and then it is uſual to place the known quantity before it.

Note. For the ſignification of this mark, $\times$ ſee introduct. at the cloſe of the 7th article. Note alſo, that this mark $=$ is a mark of equality, ſhewing that the quantities between which it ſtands, are equal to each other, and muſt be read as the ſenſe requires: thus $2\times6=3\times4=12$ may be read thus; 2×6 equals 3×4 equal to 12: or thus; 2×6 is equal to 3×4, which is equal to 12.

Examples of ſimple algebraic multiplication.

1ſt, $4ab\times5a=20aab$.	2d, $-5ab\times6bc=-30abbc$.
3d, $6ac\times-7bd=-42abcd$.	4th, $-7a\times-b=+7ab$.
5th, $x\times3x=3xx$.	6th, $-xx\times-x=+xx$.
7th, $-5ab\times+3=-15ab$.	8th, $\frac{2}{3}a\times\frac{4}{5}b=\frac{8}{15}ab$.

Diſtinctions to be obſerved betwixt addition and multiplication.

That the young algebriſt may not confound the operations of addition and multiplication, as is frequently done; I ſhall here ſet down ſome marks of diſtinction, which he ought to attend to:

As *first*, a added to a makes $2a$, but a multiplied into a makes aa.

2*dly*, a added to o makes a, but a multiplied into o makes o.

3*dly*, a added to $-a$ makes o, but a multiplied into $-a$ makes $-aa$.

4*thly*, $-a$ added to $-a$ makes $-2a$, but $-a$ multiplied into $-a$ makes $+aa$.

5*thly*, a added to 1 makes $a+1$, but a multiplied into 1 makes a.

6*thly*, $2a$ added to $-3b$ makes $2a-3b$, but $2a$ multiplied into $-3b$ makes $-6ab$.

For a further confirmation of the learner, I have added, by way of exercise in his algebraic language, the following equations; which I desire he would compute after me. Suppose $a=7$, and $b=3$: then we shall have 1st, $a+b=10$. 2dly, $a-b=4$. 3dly, $4a+5b=43$. 4thly, $4a-5b=13$. 5thly, $aa=49$. 6thly, $ab=21$. 7thly, $bb=9$. 8thly, $aaa=343$. 9thly, $aab=147$. 10thly, $abb=63$. 11thly, $bbb=27$. 12thly, $aa+2ab+bb=49+42+9=100$. 13thly, $aa-2ab+bb=49-42+9=16$. 14thly, $aaa+3aab+3abb+bbb=343+441+189+27=1000$. 15thly, $aaa-3aab+3abb-bbb=343-441+189-27=64$.

Of powers and their indexes.

7. Whenever in multiplication a letter is to be repeated oftener than once, it is usual by way of compendium, to write down the letter with a small figure after it, shewing how often that letter is to be repeated: thus instead of xx we write x^2, instead of xxx we write x^3, instead of $xxxx$ we write x^4, &c. These products are called powers of x; the figures representing the number of repetitions, are called the indexes of those powers; and the quantity x from whence all these powers arise, is called the root of these powers, or the first power of x; x^2 is called the second power of x, x^3 the third power, x^4 the fourth power, &c. *Vieta*, *Oughtred*, and some other analysts, instead of small letters used capitals, and instead of numeral indexes, distinguished these powers by names: thus *Vieta* in particular, called x^2, *X square*; x^3, *X cube*; x^4, *X square-square*; x^5, *X square-cube*; x^6, *X cube-cube*; x^7, *X square-square-cube*, &c: which names *Oughtred* contracted, and wrote them thus; *Xq*, *Xc*, *Xqq*, *Xqc*, *Xcc*, *Xqqc*, &c. but now these names are pretty much out of use, except the two first, when applied to a line squared or cubed.

If we suppose $x=5$, we shall have $2x=10$, $x^2=25$, $3x=15$, $x^3=125$, $4x=20$, $x^4=625$, &c.

The multiplication of these powers is easy: thus $x^2 \times x^3 = x^5$, because $xx \times xxx = xxxxx$: whence it may be observed, that the addition of indexes will always answer to the multiplication of powers, provided they be powers of the same quantity; for as $2+3=5$, so $x^2 \times x^3 = x^5$, &c: but

but if they be powers of different quantities, their indexes muſt not be added: thus $a^2 \times x^3 = a^2 x^3$, and $a^2 x^3 \times a^4 x^5 = a^6 x^8$. And here it muſt be obſerved, that if a number be found between two letters, it muſt always be referred to the former letter: thus $a^2 x^3$ does not ſignifie $a \times 2x^3$, but $a^2 \times x^3$.

The multiplication of ſurds.

8. This mark $\surd$ ſignifies the ſquare root of the number to which it is prefixed, and is generally prefixed to numbers whoſe ſquare root cannot be otherwiſe expreſſed, either by whole numbers or fractions: thus $\surd 2$ ſignifies the ſquare root of 2; $\surd a$ the ſquare root of *a*, &c. Theſe roots are commonly called ſurd roots, or irrational roots, becauſe their proportion to unity cannot be expreſſed in numbers.

Whenever two ſurd numbers are to be multiplied together, the ſhorteſt way will be, to multiply the numbers themſelves one into the other without any regard to the radical ſign, and then to prefix the radical ſign to the product. Thus if $\surd a$ is to be multiplied into $\surd b$, the product will be $\surd ab$; which I thus demonſtrate: let $\surd a = x$, and $\surd b = y$; then will $x^2 = a$, and $y^2 = b$, and $x^2 y^2 = ab$, and $xy = \surd ab$; but xy, or $x \times y = \surd a \times \surd b$ by the ſuppoſition; therefore, $\surd a \times \surd b = \surd ab$. Thus $\surd 2 \times \surd 3 = \surd 6$.

Theſe multiplications are of conſiderable uſe not only in matters of ſpeculation, but alſo in practice: for ſuppoſe I had occaſion to multiply the ſquare root of 2 into the ſquare root of 3, if I had not this rule, I muſt firſt extract the root of 2, to what degree of exactneſs I think proper for my purpoſe; then again I muſt extract the root of 3 to the ſame degree of exactneſs; and laſtly I muſt multiply theſe two roots together, before I can obtain the number wanted: but after it is known that $\surd 2 \times \surd 3 = \surd 6$, the whole operation will then be reduced to the extraction of the root of 6 only: nay it ſometimes happens, that two roots, though both irrational, ſhall have a rational product: thus $\surd 2 \times \surd 8 = \surd 16 = 4$, and $\surd ab^2 \times \surd ac^2 = \surd a^2 b^2 c^2 = abc$.

Of the multiplication of compound algebraic quantities.

9. The multiplication of compound algebraic quantities is performed, firſt by multiplying the multiplicand into every particular member of the multiplicator, and then reducing the whole product into the leaſt compaſs poſſible.

As for example; let it be required to multiply this compound quantity $6x - 7a - 8b$ into this compound quantity $2x - 3a + 4b$: here having

having put down the multiplicand, and the multiplicator under it, and beginning at the left hand, (for it is all one which way the operation is carried on,) I multiply the whole multiplicand into $2x$, the firſt member of my multiplicator, and the product is $12xx - 14ax - 16bx$, which I put down: then I multiply the multiplicand into $-3a$, the next member of the multiplicator, and the product is $-18ax + 21aa + 24ab$; whereof the firſt member $-18ax$, I place under $-14ax$ before found, being of the ſame denomination, for the conveniency of adding; the reſt, to wit, $+21aa + 24ab$, I place in the firſt line: this done, I now multiply by $4b$, the laſt member of the multiplicator, and the product is $24bx - 28ab - 32bb$; whereof I place $24bx$ under $-16bx$, and $-28ab$ under $+24ab$, and the laſt member $-32bb$ I place in the firſt line, as having no quantity of the ſame denomination to join with it: laſtly I reduce the whole product into the leaſt compaſs poſſible; and it ſtands thus: $12xx - 32ax + 8bx + 21aa - 4ab - 32bb$. See the work:

$$\begin{array}{llllll}
 & 6x & -7a & -8b & & \\
 & 2x & -3a & +4b & & \\
\hline
 & 12xx & -14ax & -16bx + 21aa & +24ab & -32bb \\
 & & -18ax & +24bx & -28ab & \\
\hline
\text{Sum} & 12xx & -32ax & +8bx + 21aa & -4ab & -32bb.
\end{array}$$

Example 2d.

$$\begin{array}{l}
3x + 4a - 5b \\
3x - 4a + 5b \\
9xx + 12ax - 15bx - 16aa + 20ab - 25bb \\
\quad - 12ax + 15bx \qquad + 20ab \\
\hline
9xx \quad * \quad * \quad - 16aa + 40ab - 25bb.
\end{array}$$

Example 3d.

$$\begin{array}{l}
6xx - 7ax + 8aa \\
2xx - 3ax + 4aa \\
\hline
12x^4 - 14ax^3 + 16a^2x^2 - 24a^3x + 32a^4 \\
\quad - 18ax^3 + 21a^2x^2 - 28a^3x \\
\qquad + 24a^2x^2 \\
\hline
12x^4 - 32ax^3 + 61a^2x^2 - 52a^3x + 32a^4.
\end{array}$$

Example

Example 4th.

$$\begin{array}{l} a+b \\ a-b \\ \hline aa+ab-bb \\ \quad -ab \\ \hline aa \quad * -bb. \end{array}$$

Example 5th.

$$\begin{array}{l} a+b \\ a+b \\ \hline aa+ab+bb \\ \quad +ab \\ \hline aa+2ab+bb. \end{array}$$

Example 6th.

$$\begin{array}{l} a-b \\ a-b \\ \hline aa-ab+bb \\ \quad -ab \\ \hline aa-2ab+bb. \end{array}$$

N. B. A dash over two or more quantities, signifies that all those quantities are to be taken into one conception, or to be considered as making up but one compound quantity: thus $\overline{a+b}\times\overline{c-d}$ does not signify that which arises from multiplying $b\times c$, and then adding $a-d$ to the product, as it might be mistaken without the dash; but it signifies the product of the whole quantity $\overline{a+b}$ multiplied into the whole quantity $\overline{c-d}$.

The proof of compound multiplication.

10. In the third example we multiplied $\overline{6xx-7ax+8aa}$ into $\overline{2xx-3ax+4aa}$, and the product amounted to $12x^4-32ax^3+61a^2x^2-52a^3x+32a^4$: let us try this in numbers, and see how it will answer. In order to which, we may suppose a and x equal to any two numbers whatever, but the simplest way of tryal will be to make a equal 1, and $x=1$; and then we shall have in the multiplicand $6xx=6$, $-7ax=-7$, and $+8aa=+8$, and $6-7+8=7$; therefore the multiplicand is 7: again, in the multiplicator we have $2xx=2$, $-3ax=-3$, $+4aa=+4$, and $2-3+4=3$; therefore the multiplicator is 3: and 7 the multiplicand, multiplied into 3 the multiplicator, gives

21 for the product. Let us now examine the ſeveral parts of the product, as they are here repreſented in letters, and ſee whether they will amount to that number: $12x^4=12$, $-32ax^3=-32$, $+61a^2x^2=+61$, $-52a^3x=-52$, $+32a^4=+32$; and $12-32+61-52+32$ amount to juſt 21. This may ſerve as a proof to the work, though not a neceſſary one; for it is not impoſſible but there may be a conſiſtency this way, and yet the work be falſe; but this will rarely happen, unleſs it be deſigned. But the work may ſtill be confirmed by making $a=1$, and $x=-1$; for then the multiplicand will be $6+7+8=21$; and the multiplicator $2+3+4=9$; and the product $12+32+61+52+32=189$, which is the ſame with the product of 21 the multiplicand, multiplied into 9 the multiplicator.

How general theorems may be obtained by multiplication in Algebra.

11. From theſe algebraic multiplications are derived and demonſtrated many very uſeful theorems in all the parts of Mathematics; whereof I ſhall juſt give the learner a taſte, and then proceed to another ſubject.

In the fourth example of compound multiplication we found, that $a+b$ multiplied into $a-b$ produced $aa-bb$; whence I infer, that *The ſum and difference of any two numbers multiplied together will give the difference of their ſquares, and* vice versâ: for a and b will repreſent any two numbers at pleaſure; $a+b$ their ſum, $a-b$ their difference, and $aa-bb$ the difference of their ſquares: thus if we aſſume any two numbers whatever, ſuppoſe 7 and 3, the difference of their ſquares is $49-9$, or 40; and 10 their ſum, multiplied into 4 their difference, makes alſo 40.

But here I am to give notice once for all, that inſtances in numbers ſerve well enough to illuſtrate a general theorem, but they muſt not by any means be looked upon as a proof of it; becauſe a propoſition may be true in ſome particular caſes inſtanced in, and yet fail in others; but whenever a propoſition is found to be true *in ſpeciebus*, that is, in letters or ſymbols, it is a ſufficient demonſtration of it, becauſe theſe are univerſal repreſentations.

In the 5th example it is ſhewn, that $a+b$ multiplied into itſelf produced $aa+2ab+bb$; whence I infer, that *If a number be reſolved into any two parts whatever, the ſquare of the whole will be equal to the ſquare of each part, and the double rectangle, or product of the multiplication of thoſe parts, added together:* thus if the number 10 be reſolved into 7 and 3; 100 the ſquare of 10, the whole, will be equal to 49 the ſquare of 7, and

and 9 the ſquare of 3, and 42 the double product of 7 and 3 multiplied together: for $49+9+42=100$.

In the 6th example we found, that $a-b$ multiplied into itſelf, produced $aa-2ab+bb$; whence I infer, that *If from the ſum of the ſquares of any two numbers, be ſubtracted the double product of thoſe numbers, there will remain the ſquare of their difference:* for $aa+bb$ is the ſum of the ſquares of a and b, and $2ab$ is their double product, and $aa-2ab+bb$ was found to be the ſquare of $a-b$, that is, the ſquare of the difference of a and b: thus in the numbers 7 and 3, the ſquare of 7 is 49, the ſquare of 3 is 9, and the ſum of their ſquares is 58; and if from this be ſubtracted the double product 42, the remainder will be 16, the ſquare of 4, that is, the ſquare of the difference of the numbers 7 and 3.

Theſe two laſt theorems are in ſubſtance the fourth and ſeventh propoſitions of the ſecond book of *Euclid.*

How to expreſs the three ſides of a right-angled triangle in rational numbers.

12. From theſe two laſt theorems may be ſolved a problem of no ſmall eſtimation among Algebriſts; which is, to find three numbers, that ſhall repreſent the three ſides of a right-angled triangle; or rather, to find as many ſets as we pleaſe of ſuch numbers; that is, in other words, *To find three numbers of ſuch a nature, that the ſum of the ſquares of two of them may be equal to the ſquare of the third.* Now it is plain that theſe three numbers $aa-2ab+bb$, and $4ab$, and $aa+2ab+bb$ are ſuch, that the two firſt added together, are equal to the third: it is certain alſo, that of theſe three numbers, the two extremes are ſquare numbers, that is, are ſuch as will admit of an exact ſquare root; for $aa-2ab+bb$, is the ſquare of $a-b$; and $aa+2ab+bb$ is the ſquare of $a+b$; therefore, if the middle number $4ab$ was a ſquare number like the reſt, we ſhould have 3 ſquare numbers, whereof the two firſt added together would be equal to the third; and conſequently, the roots of theſe three ſquares, would be three numbers that would anſwer the condition of the problem. But the middle number $4ab$ will be a ſquare, if a and b be ſquare numbers: for if we ſuppoſe $a=rr$, and $b=ss$, we ſhall have $4ab=4rrss$, which is a ſquare number whoſe root is $2rs$: but the root of the firſt ſquare $aa-2ab+bb$, was $a-b$, which in this caſe is $rr-ss$; and the root of the third ſquare, $aa+2ab+bb$ was $a+b$, which is $rr+ss$; therefore theſe three numbers, $rr-ss$, $2rs$ and r^2+s^2 are ſuch, that the ſquare of the firſt added to the ſquare of the ſecond will make the ſquare of the third; and this will be the caſe,

whatever numbers r and s are made to ſtand for. But $rr - ss$ is the difference of the ſquares of r and s, and $2rs$ is the double product of their multiplication, and laſtly $rr + ss$ is the ſum of the ſquares of r and s. Since then we are at liberty to aſſume what numbers we pleaſe for r and s, the problem will receive the following ſolution: *Take any two numbers at pleaſure, and from theſe two numbers thus taken, derive three others thus: take the difference of their ſquares, the double product of their multiplication, and the ſum of their ſquares, and the three numbers thus found will anſwer the condition of the problem.* As for example, let the numbers taken be 2 and 1: now the difference of the ſquares of theſe numbers is $4 - 1 = 3$; the double product of their multiplication is $2 \times 2 \times 1 = 4$; and the ſum of their ſquares is $4 + 1 = 5$; therefore the numbers 3, 4 and 5 are ſuch as will anſwer the condition propoſed: and ſo we find them; for $3 \times 3 + 4 \times 4 = 5 \times 5$, that is, $9 + 16 = 25$. Again, let the numbers aſſumed be 3 and 2; and the difference of their ſquares will be $9 - 4 = 5$, their double product $2 \times 3 \times 2 = 12$, and the ſum of their ſquares $9 + 4 = 13$; therefore 5, 12 and 13 is another ſet of numbers that will anſwer the condition of the problem: for $5 \times 5 + 12 \times 12 = 13 \times 13$, that is, $25 + 144 = 169$. Laſtly, let 4 and 1 be the numbers aſſumed; and the difference of their ſquares will be $16 - 1 = 15$, their double product $2 \times 4 \times 1 = 8$, and the ſum of their ſquares $16 + 1 = 17$; therefore 8, 15 and 17, will alſo anſwer the condition of the problem: for $8 \times 8 + 15 \times 15 = 17 \times 17$, that is, $64 + 225 = 289$.

Of the diviſion of ſimple algebraic quantities.

13. The diviſion of ſimple algebraic quantities, where it is poſſible in integral terms, is performed, firſt by dividing the numeral coefficient of the dividend by the numeral coefficient of the diviſor, and then putting down after the quotient, all the letters in the dividend, that are not in the diviſor; the ſign of the quotient in diviſion being determined by thoſe of the diviſor and dividend, juſt in the ſame manner as the ſign of the product in multiplication is determined by thoſe of the multiplicator and multiplicand; that is, if the ſigns of the diviſor and dividend be both alike, whether they be both affirmative, or both negative, the quotient will be affirmative, otherwiſe it will be negative: thus if the quantity $-12ab$ is divided by $-3a$, the quotient will be $+4b$; which I thus demonſtrate: In all diviſion whatever, the quotient ought to be ſuch a quantity, as being multiplied by the diviſor, will make the dividend; therefore, to enquire for the quotient in our caſe, is nothing elſe, but to enquire what number, or quantity, multiplied into $-3a$, the diviſor,

visor, will produce $-12ab$, the dividend. First then I ask, what sign multiplied into $-$, the sign of the divisor, will give $-$ the sign of the dividend, and the answer is $+$; therefore $+$ is the sign of the quotient: in the next place I enquire, what number multiplied into 3, the coefficient of the divisor, will give 12, the coefficient of the dividend, and the answer is 4; therefore 4 is the coefficient of the quotient: lastly I enquire, what letter multiplied into a, the letter of the divisor, will produce ab, the denominator, or literal part of the dividend, and the answer is b; therefore b is the letter of the quotient: and thus at last we have the whole quotient, which is $+4b$. And this way of reasoning will carry the learner through all the other cases.

Examples of simple division in Algebra.

Example 1st, $4ab$) $24abbc$ ($6bc$.
2d, $+7$) $-35ab$ ($-5ab$.
3d, $-x$) $-3xx$ ($+3x$.
4th, $-9ab$) $+72ab$ (-8.
5th, $-4a^3$) $-60a^8$ ($+15a^5$.
6th, $4x^2$) $60x^9$ ($+15x^7$.
7th, $+4a^3x^2$) $-60a^8x^9$ ($-15a^5x^7$.
8th, b) $\frac{3}{4}ab$ ($\frac{3}{4}a$.
9th, $\frac{2}{3}$) $\frac{4}{5}b$ ($\frac{6}{5}b$.

Of the notation of algebraic fractions.

Whenever a division according to the foregoing method is found impossible, the quotient cannot be otherwise expressed than by a fraction, whose numerator is the dividend, and denominator the divisor; see the introduction, art. 13. As if it was required to divide a by b, which division is impossible according to the foregoing rule, the quotient must be expressed by this fraction $\frac{a}{b}$, which is usually read thus a by b, that is, a divided by b, or the quotient of a divided by b: for in Algebra the word *by*, is generally speaking, appropriated to division, as the word *into* is to multiplication.

If the numerator, or denominator, or both, be compound quantities, the respective fractions must be written thus; $\frac{a+b}{c}$, $\frac{a}{b+c}$, $\frac{a+b}{c-d}$.

If a division be partly possible according to the foregoing rules, and partly impossible, it must be pursued as far as it is possible, and the rest

must be represented by a fraction, as in common division: thus if $ad+bd+c$ was to be divided by d, the quotient would be $a+b+\frac{c}{d}$.

Of the division of compound algebraic quantities.

14. The division of compound algebraic quantities is performed, first, by ranging the several members both of the divisor and dividend according to the dimensions of some letter common to them both, and then proceeding as in common arithmetick.

N. B. A quantity is said to be disposed according to the dimensions of any letter in it, when the highest power of that letter is placed first, and the next in order, and so on, as in the following example; where both the divisor and dividend are ranged according to the dimensions of the letter x.

Of this compound division, take the following example: let it be required to divide this quantity $48x^3-76ax^2-64a^2x+105a^3$ by this quantity $2x-3a$. Here, as my divisor consists of two members, I take the two first members of the dividend for a first dividual; then I divide the first member of the dividual by the first member of the divisor, to wit, $48x^3$ by $2x$, and the quotient is $24xx$, which I put down in the quotient: this done, I multiply the divisor $2x-3a$ by the quotient $24xx$, and the product is $48x^3-72ax^2$, which I place under my first dividual $48x^3-76ax^2$, and then subtracting the former from the latter, I find the remainder to be $-4ax^2$; to this remainder I bring down the next place of my dividend, which is $-64a^2x$, and so have a second dividual, to wit, $-4ax^2-64a^2x$: here again I divide the first member of this dividual, by the first member of the divisor, *viz.* $-4ax^2$ by $2x$, and the quotient is $-2ax$, which I put down in the quotient; then multiplying the divisor $2x-3a$, by this last quotient $-2ax$, the product is $-4ax^2+6a^2x$, which being subtracted from the second dividual $-4ax^2-64a^2x$ leaves $-70a^2x$ for a remainder; to this remainder I bring down $+105a^3$ the last place in the dividend, and so have a third dividual $-70a^2x+105a^3$; the first member whereof divided by the first member of the divisor, quotes $-35aa$, which being put down in the quotient, and the divisor multiplied by it, the product is $-70a^2x+105a^3$, which being subtracted from the last dividual leaves no remainder; so that the whole quotient at last amounts to $24xx-2ax-35aa$. See the work:

Example

Example 1st.

$2x-3a\,)\;48x^3-76ax^2-64a^2x+105a^3\;(\,24x^2-2ax-35a^2$

$48x^3-72ax^2$

$*\quad -4ax^2-64a^2x$

$-4ax^2+6a^2x$

$*\quad -70a^2x+105a^3$

$-70a^2x+105a^3$

$*\quad *$

Example 2d.

$4x-5a\,)\;48x^3-76ax^2-64a^2x+105a^3\;(\,12x^2-4ax-21aa$

$48x^3-60ax^2$

$*\quad -16ax^2-64a^2x$

$-16ax^2+20a^2x$

$*\quad -84a^2x+105a^3$

$-84a^2x+105a^3$

$*\quad *$

Example 3d.

$6x+7a\,)\;48x^3-76ax^2-64a^2x+105a^3\;(\,8x^2-22ax+15a^2$

$48x^3+56ax^2$

$*\quad -132ax^2-64a^2x$

$-132ax^2-154a^2x$

$*\quad +90a^2x+105a^3$

$+90a^2x+105a^3$

$*\quad *$

Example

Example 4th.

$$
\begin{array}{l}
3x^2-4x+5\;)\;18x^4-45x^3+82x^2-67x+40\;(\;6x^2-7x+8\\
\qquad\qquad\quad\;\; \underline{18x^4-24x^3+30x^2}\\
\qquad\qquad\qquad * \;\; -21x^3+52x^2-67x\\
\qquad\qquad\qquad\quad\;\; \underline{-21x^3+28x^2-35x}\\
\qquad\qquad\qquad\qquad\quad * \;\; +24x^2-32x+40\\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{+24x^2-32x+40}\\
\qquad\qquad\qquad\qquad\qquad\quad * \quad * \quad *
\end{array}
$$

This, as I take it, is the most intelligible way of working these divisions, as being the nearest of kin to common division; but it may be somewhat contracted, as will easily be seen by working the foregoing examples over again thus:

Example 1st.

$$
\begin{array}{l}
2x-3a\;)\;48x^3-76ax^2-64a^2x+105a^3\;(\;24x^2-2ax-35aa\\
\qquad\qquad\; \underline{48x^3-72ax^2}\\
\qquad\qquad\; * \quad -4ax^2\\
\qquad\qquad\quad\;\; \underline{-4ax^2+6a^2x}\\
\qquad\qquad\qquad\qquad -70a^2x\\
\qquad\qquad\qquad\qquad \underline{-70a^2x+105a^3}\\
\qquad\qquad\qquad\qquad\quad * \qquad *
\end{array}
$$

Example 2d.

$$
\begin{array}{l}
4x-5a\;)\;48x^3-76ax^2-64a^2x+105a^3\;(\;12x^2-4ax-21a^2\\
\qquad\qquad\; \underline{48x^3-60ax^2}\\
\qquad\qquad\; * \;\; -16ax^2\\
\qquad\qquad\quad\;\; \underline{-16ax^2+20a^2x}\\
\qquad\qquad\qquad * \;\; -84a^2x\\
\qquad\qquad\qquad\quad\;\; \underline{-84a^2x+105a^3}\\
\qquad\qquad\qquad\qquad\quad * \qquad *
\end{array}
$$

Example

Example 3d.

$$6x+7a\ )\ 48x^3-76ax^2-64a^2x+105a^3\ (\ 8x^2-22ax+15aa$$
$$48x^3+56ax^2$$
$$*\ -132ax^2$$
$$-132ax^2-154a^2x$$
$$*\quad +90a^2x$$
$$+90a^2x+105a^3$$
$$*\quad *$$

Example 4th.

$$3x^2-4x+5\ )\ 18x^4-45x^3+82x^2-67x+40\ (\ 6x^2-7x+8$$
$$18x^4-24x^3+30x^2$$
$$*\ -21x^3+52x^2$$
$$-21x^3+28x^2-35x$$
$$*\quad +24x^2-32x$$
$$+24x^2-32x+40$$
$$*\quad *\quad *$$

This sort of division may be proved the same way as in common division, to wit, by multiplying the divisor into the quotient, or the quotient into the divisor, and adding the remainder, if there be any; for then, if the product, or sum, be equal to the dividend, the division is right, otherwise not: thus if instead of dividing $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$ as in the first example, we divide $48x^3-76ax^2-64a^2x+110a^3$ by $2x-3a$, the quotient will be the same as before, to wit, $24x^2-2ax-35a^2$, but then there will be a remainder of $5a^3$; therefore *e converso*, if the quotient $24x^2-2ax-35a^2$ be multiplied into the divisor $2x-3a$, the product together with the remainder $5a^3$ will make the dividend $48x^3-76ax^2-64a^2x+110a^3$. See the proof:

$$\begin{array}{l}
24x^2 - 2ax - 35a^2 \\
\;\; 2x - 3a \\
\hline
48x^3 - 4ax^2 - 70a^2x + 105a^3 \\
\qquad\;\; - 72ax^2 + 6a^2x \\
\hline
48x^3 - 76ax^2 - 64a^2x + 105a^3 \\
\qquad\qquad\qquad\qquad\qquad\quad + 5a^3 \\
\hline
48x^3 - 76ax^2 - 64a^2x + 110a^3.
\end{array}$$

If, when the divisor and dividend are placed according to the dimensions of some common letter, there be any places wanting, these may be supplied by stars: as if it was required to divide $16x^4 - 72a^2x^2 + 81a^4$ by $2x - 3a$; there are wanting two places in the dividend, to wit, the places where the third and simple powers of x are concerned: these therefore being supplied, the dividend will stand thus; $16x^4 \ast - 72a^2x^2 \ast + 81a^4$. See the work:

$$\begin{array}{l}
2x - 3a)\; 16x^4 \quad \ast \quad - 72a^2x^2 \quad \ast + 81a^4 \;(8x^3 + 12ax^2 - 18a^2x - 27a^3 \\
\qquad\qquad 16x^4 - 24ax^3 \\
\hline
\qquad\qquad\;\; \ast \;\; + 24ax^3 \\
\qquad\qquad\qquad + 24ax^3 - 36a^2x^2 \\
\hline
\qquad\qquad\qquad\quad \ast \quad - 36a^2x^2 \\
\qquad\qquad\qquad\qquad - 36a^2x^2 + 54a^3x \\
\hline
\qquad\qquad\qquad\qquad\qquad \ast \quad - 54a^3x \\
\qquad\qquad\qquad\qquad\qquad\quad - 54a^3x + 81a^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad \ast \qquad \ast
\end{array}$$

For another example of this kind, let it be required to divide $81x^4 - 256a^4$ by $3x + 4a$, and the work will stand thus:

$$\begin{array}{l}
3x + 4a)\; 81x^4 \quad \ast \quad \ast \quad \ast - 256a^4 (27x^3 - 36ax^2 + 48a^2x - 64a^3 \\
\qquad\qquad 81x^4 + 108ax^3 \\
\hline
\qquad\qquad\;\; \ast \;\; - 108ax^3 \\
\qquad\qquad\quad - 108ax^3 - 144a^2x^2 \\
\hline
\qquad\qquad\qquad\quad \ast \quad + 144a^2x^2 \\
\qquad\qquad\qquad\qquad + 144a^2x^2 + 192a^3x \\
\hline
\qquad\qquad\qquad\qquad\qquad \ast \quad - 192a^3x \\
\qquad\qquad\qquad\qquad\qquad\quad - 192a^3x - 256a^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad \ast \qquad \ast
\end{array}$$

As

As division in decimal fractions may be often, by the help of cyphers, continued at pleasure, so likewise may division in Algebra by the help of stars: as if 1 was to be divided by $1+x$, the quotient would be $1-x+x^2-x^3+x^4-x^5$ &c *ad infinitum*: but if 1 be divided by $1-x$, the quotient will be $1+x+x^2+x^3+x^4+x^5$ &c *ad infinitum*. See the work:

$1+x)\ 1\ *\ *\ *\ *\ *\ (1-x+x^2-x^3+x^4-x^5$ &c.

$1+x$

$*-x$

$-x-x^2$

$*+x^2$

$+x^2+x^3$

$*-x^3$

$-x^3-x^4$

$*+x^4$

$+x^4+x^5$

$*-x^5.$

$1-x)\ 1\ *\ *\ *\ *\ *\ (1+x+x^2+x^3+x^4+x^5$ &c.

$1-x$

$*+x$

$+x-x^2$

$*+x^2$

$+x^2-x^3$

$*+x^3$

$+x^3-x^4$

$*+x^4$

$+x^4-x^5$

$*+x^5.$

Quotients arising from this sort of division are for the most part so regular, that a few terms may be sufficient to discover the whole process or continuation, as well as if the work was continued *ad infinitum*. As who that sees this work, and four or five of the first members of the quotient, can doubt of the rest?

For another example of this kind of diviſion, let it be required to divide unity by the quantity $1-2x+xx$, and the work will be as follows:

$$
\begin{array}{l}
1-2x+xx\ )\ 1\quad *\quad *\quad *\quad *\ (\,1+2x+3xx+4x^3+5x^4\ \&c.\\
\qquad\qquad\quad \underline{1-2x+xx}\\
\qquad\qquad\quad *+2x-xx\\
\qquad\qquad\quad \underline{\ \ +2x-4xx+2x^3}\\
\qquad\qquad\qquad\quad *+3xx-2x^3\\
\qquad\qquad\qquad\quad \underline{\ \ +3xx-6x^3+3x^4}\\
\qquad\qquad\qquad\qquad\quad *+4x^3-3x^4\\
\qquad\qquad\qquad\qquad\quad \underline{\ \ +4x^3-8x^4}\\
\qquad\qquad\qquad\qquad\qquad\quad *+5x^4\\
\qquad\qquad\qquad\qquad\qquad\qquad +5x^4.
\end{array}
$$

1ſt, By the four ſtars annexed to the dividend unity, is ſhewn, that this diviſion, though productive of an infinite ſeries, is not here intended to be continued beyond the fourth power of x.

2dly, By the regularity of the quotient $1+2x+3xx+4x^3+5x^4$, none can doubt but that the next ſubſequent terms will be $+6x^5+7x^6+8x^7$ *&c ad infinitum.*

3dly, When quotients are thus regular, the very remainders will be ſo too: thus in the preſent caſe, the remainders are $2x-1x^2$, $3x^2-2x^3$, $4x^3-3x^4$, $5x^4-4x^5$; but of this laſt remainder, the latter part $-4x^5$ is dropped, as being out of the compaſs of the work; and ſo muſt all other terms be that are ſo, whether it be in multiplication, diviſion, extraction of roots, or in any other operation whatever.

Let us now try the foregoing diviſion by multiplication; that is, let the quotient $1+2x+3x^2+4x^3+5x^4$ be multiplied by the diviſor $1-2x+xx$, and let us ſee whether the product will be $1\ *\ *\ *\ *$; for as the quotient is not true to above five places, it muſt not be expected that the product ſhould be true to more. See the work, where the multiplication is carried on from left to right:

$$
\begin{array}{l}
1+2x+3x^2+4x^3+5x^4\\
\underline{1-2x+x^2\qquad\qquad\qquad}\\
1+2x+3x^2+4x^3+5x^4\\
\ \ -2x-4x^2-6x^3-8x^4\\
\qquad\quad\ \underline{+x^2+2x^3+3x^4}\\
1\quad *\quad *\quad *\quad *.
\end{array}
$$

There

There is alſo another way of performing theſe multiplications, which I ſhall here put down; not ſo much for any extraordinary uſe it has in theſe caſes, as that it may ſerve very well to explain Mr. *Oughtred*'s rule for contracting multiplication, which ſee in his *Clavis*, chap. 4.

Here then the multiplicator muſt be inverted, wherein the place of units muſt be put under that power of the multiplicand which is intended to be the laſt in the product; that is, in the preſent caſe, 1 muſt be placed under $5x^4$, thus:

$$\begin{array}{rrrrr} 1 & +2x & +3x^2 & +4x^3 & +5x^4 \\ & & x^2 & -2x & +1. \end{array}$$

Having thus put down the multiplicand and multiplicator, every term of the multiplicator muſt begin to multiply that term of the multiplicand which is next above it, and then muſt proceed to multiply all the other terms towards the left hand: thus 1 muſt multiply $5x^4+4x^3+3x^2+2x+1$; but the next term of the multiplicator, to wit, $-2x$ muſt multiply only $4x^3+3x^2+2x+1$; and the laſt term of the multiplicator, to wit, xx, muſt multiply only $3xx+2x+1$, as may be ſeen by the work, which follows:

$$\begin{array}{rrrrr} 1 & +2x & +3x^2 & +4x^3 & +5x^4 \\ & & x^2 & -2x & +1 \\ \hline 1 & +2x & +3x^2 & +4x^3 & +5x^4 \\ & -2x & -4x^2 & -6x^3 & -8x^4 \\ & & x^2 & +2x^3 & +3x^4 \\ \hline 1 & * & * & * & *. \end{array}$$

N. B. In the application of this example to *Oughtred*'s rule, x muſt be made equal to $\frac{1}{10}$, xx to $\frac{1}{100}$, &c.

I could not well ſay leſs than I have done concerning diviſion, to give the young Algebriſt a tolerable notion of that operation; but to ſay more, and to enter into a detail of all the particular caſes that may happen in literal diviſion, would be to fill his head full of abſtracted notions, which he is almoſt ſure to forget, before ever he will come to apply them. It is for the ſame reaſon that I ſhall at this time paſs by the invention of diviſors, the doctrine of ſurds, the reduction of radicals, and ſome other things of the ſame nature, till we ſhall have a more immediate demand for them, and then I ſhall produce and explain them as we want them, without obſerving any other method.

Of Proportion in numbers.

15. The rule of proportion in Algebra is ſo very little different from the rule of proportion in common arithmetick, that one example of it will be ſufficient. Let then the following queſtion be put: *If* a *gives* b, *what will* c *give?* here the ſecond and third terms multiplied together produce bc, and the quotient of this divided by the firſt term a, cannot otherwiſe be expreſſed than by the fraction $\frac{bc}{a}$: this is evident from the notation of fractions explained in the 13th article. But as I have hitherto purpoſely avoided all conſideration of proportion, chooſing rather to appeal upon all occaſions, to the common idea every one has or thinks he has of it, than to be more particular, it may not be improper, now we come to reaſon more cloſely upon things, to enter more diſtinctly into the particular nature of proportion, ſo far at leaſt as it relates to numbers, and ſhew wherein it conſiſts.

According to *Euclid*, four numbers are ſaid to be proportionable, that is, the firſt number is ſaid to have the ſame proportion to the ſecond, that the third hath to the fourth; or the firſt is ſaid to be to the ſecond, as the the third is to the fourth, when the firſt number is the ſame multiple, part or parts, of the ſecond, that the third is of the fourth: but it will be aſked perhaps; How can we know, what parts, part, or multiple, any one number is of another? To which I anſwer by a fraction, whoſe numerator is the former number, and denominator the latter: thus the fraction $\frac{2}{3}$ expreſſly ſhews, that the numerator 2 is two third parts of the denominator 3; for this is certain, that 1 is $\frac{1}{3}$ part of 3, and therefore 2 muſt be $\frac{2}{3}$ of it: for the ſame reaſon the fraction $\frac{12}{8}$ ſhews that the number 12 is $\frac{12}{8}$ or $\frac{3}{2}$ of the number 8; and laſtly, the fraction $\frac{12}{4}$ ſhews, that the number 12 is $\frac{12}{4}$ of, or 3 times the number 4, and conſequently, that 12 is a multiple of 4, as containing it juſt 3 times without any remainder: therefore to any one who underſtands fractions, *Euclid*'s definition of proportion may be more diſtinctly expreſſed thus: *Four numbers are ſaid to be proportionable, when a fraction whoſe numerator is the firſt number, and denominator the ſecond, is equal to a fraction whoſe numerator is the third number and denominator the fourth.* Thus 2 is to 3 as 4 is to 6, becauſe $\frac{2}{3}$ is equal to $\frac{4}{6}$; thus 12 is to 8 as 15 is to 10, becauſe $\frac{12}{8}$ equals $\frac{15}{10}$, both being reducible to $\frac{3}{2}$; thus 2 is to 6 as 4 is to 12, becauſe $\frac{2}{6}$ equals $\frac{4}{12}$, for each is equal to $\frac{1}{3}$; laſtly, 6 is to 2 as 12 is to 4, becauſe $\frac{6}{2}=\frac{12}{4}=3$.

From this idea of proportionality may be demonſtrated a very uſeful theorem in Algebra; which is, that *Whenever four numbers are proportionable,*

the

the product of the extreme terms multiplied together will be equal to the product of the two middle terms so multiplied: for let a, b, c and d, be four proportionable numbers in their order; that is, let a be to b as c is to d; I say then that ad the product of the extremes will be equal to bc the product of the two middle terms: for since a is to b as c is to d, it follows from what has already been laid down, that the fraction $\frac{a}{b}$ is equal to the fraction $\frac{c}{d}$; multiply both the terms of the fraction $\frac{a}{b}$ into d, and both those of the fraction $\frac{c}{d}$ into b, (which multiplications may be made without altering the values of the fractions,) and then you will have $\frac{ad}{bd}=\frac{bc}{bd}$; that is, the quotient of ad divided by bd, is equal to the quotient of bc divided by bd; therefore ad must be equal to bc, that is, the product of the extremes must be equal to the product of the middle terms. *Q. E. D.*

The converse of this proposition is also true, to wit, that *Whenever we have an equation in numbers, wherein the product of two numbers on one side is found equal to the product of two numbers on the other, such an equation may be resolved into four proportionals, by making the two numbers on either side, the extremes, and those on the other side, the middle terms:* thus if $ad=bc$; by making a and d the extremes, and b and c the middle terms, we shall have a to b as c to d: if this be denied, let a be to b as c is to e; then we shall have $ae=bc$ by the last; but $ad=bc$ by the supposition; therefore $ae=ad$; therefore e equals d, and a is to b as c is to d. *Q. E. D.*

COROLLARY.

Whence if a, b and c, be continual proportionals, that is, if a is to b as b is to c, we shall have $b^2=ac$: and *e converso*, if $b^2=ac$, then a, b and c will be continual proportionals.

The common properties of proportionality in numbers demonstrated.

16. From what has been delivered in the last article, may be demonstrated all or most of the common properties of proportionable numbers with a great deal of ease, some of the most useful whereof I shall here throw

throw together into one ſingle article, for the reader to peruſe, either at preſent, or hereafter, as he ſhall ſee occaſion.

Firſt then, from what has been ſaid, may the rule of three, which conſiſts in finding a fourth proportional, be moſt diſtinctly demonſtrated: for let a, b and c be three numbers given, in order to find d, a fourth proportional; then ſince a is to b as c is to d, you will have ad the product of the extremes, equal to bc the product of the middle terms; divide both ſides of the equation by a, and you will have $d = \frac{bc}{a}$; which is as much as to ſay, that if three numbers be given, a fourth proportional may be obtained by multiplying the ſecond and third numbers together, and dividing the product by the firſt.

In the rule of three inverſe, let the numbers when diſpoſed according to form be a, b and c; then whoſoever attentively conſiders the nature of that rule, will eaſily ſee, that the fourth number there ſought for, is not to be a fourth proportional to the three numbers given as they are diſpoſed in the order a, b, c, but as they ſtand in the order c, b, a, or c, a, b, and therefore in this caſe, the fourth number will be $\frac{ab}{c}$.

Secondly, if two proportions be equal to a third, they muſt be equal to one another, becauſe if two fractions be equal to a third, they muſt be equal to one another: thus if a is to b as c is to d, and c is to d as e is to f, we ſhall have a to b as e to f.

Thirdly, if a is to b as c is to d; then b will be to a as d to c, which is called inverſe proportion: for if a is to b as c is to d, we ſhall have $ad = bc$; make b and c the extremes, and you will have b to a as d to c.

Fourthly, if a is to b as c is to d; we ſhall have by permutation, a to c as b to d: for ſince a is to b as c is to d, and conſequently $ad = bc$, make a and d the extremes, and c and b the middle terms, and you will have a to c as b to d.

Fifthly, if a is to b as c is to d, and any two multiplicators whatever be aſſumed, as e and f; I ſay then, that ea is to fb, as ec to fd: for ſince a is to b as c is to d, and ſo $ad = bc$; multiply both ſides of the equation by the product ef, and you will have $ad \times ef = bc \times ef$; but $ad \times ef = ea \times fd$, and $bc \times ef = fb \times ec$; therefore $ea \times fd = fb \times ec$; make ea and fd extremes, and the proportion will ſtand thus; ea is to fb as ec to fd. In like manner, *mutatis mutandis*, it may be demonſtrated, that if a is to b as c is to d, then $\frac{a}{e}$ will be to $\frac{b}{f}$ as $\frac{c}{e}$ is to $\frac{d}{f}$.

Sixthly, if a is to b as c is to d; then a^2 is to b^2 as c^2 is to d^2: for ſince a is to b as c is to d, and ſo $ad = bc$; ſquare both ſides of the equation,

equation, and you will have $a^2 d^2 = b^2 c^2$; make a^2 and d^2 extremes, and you will have a^2 to b^2 as c^2 to d^2. And by taking theſe ſteps backwards, it will alſo appear, that if a^2 is to b^2 as c^2 is to d^2; a is to b as c is to d, and $\sqrt{a}$ is to $\sqrt{b}$ as $\sqrt{c}$ is to $\sqrt{d}$.

Seventhly, if a is to b as c is to d; then by compoſition (as it is called) $\overline{a+b}$ is to b as $\overline{c+d}$ is to d; or $\overline{a+b}$ is to a, as $\overline{c+d}$ is to c: for ſince a is to b as c is to d, and conſequently $ad = bc$; add bd to both ſides of the equation, and you will have $ad+bd = bc+bd$; but $ad+bd$ is the product of $\overline{a+b}$ multiplied into d, as is eaſily ſeen; and $bc+bd$ is the product of b multiplied into $\overline{c+d}$; therefore $\overline{a+b} \times d = b \times \overline{c+d}$; make $\overline{a+b}$ and d extremes, and you will have $\overline{a+b}$ to b as $\overline{c+d}$ to d. Again, ſince $bc = ad$, add ac to both ſides, and you will have $ac+bc = ac+ad$, that is, $\overline{a+b} \times c = a \times \overline{c+d}$; make $\overline{a+b}$ and c extremes, and you will have $\overline{a+b}$ to a as $\overline{c+d}$ to c.

Eighthly, if a is to b as c is to d; then by diviſion $\overline{a-b}$ is to b as $\overline{c-d}$ is to d; or $\overline{a-b}$ is to a as $\overline{c-d}$ to c. This propoſition is demonſtrated by ſubtraction, juſt in the ſame manner as the laſt was by addition.

Ninthly, if to or from two numbers in any given proportion, be added or ſubtracted other two numbers in the ſame proportion, the ſums or remainders will ſtill be in the ſame proportion with the numbers firſt propoſed: thus if the numbers c and d be in the ſame proportion with the numbers a and b, that is, if as a is to b ſo is c to d, and if to or from the former two numbers, be added or ſubtracted the latter, we ſhall have not only $\overline{a+c}$ to $\overline{b+d}$ as a to b, but alſo $\overline{a-c}$ to $\overline{b-d}$ as a to b: for ſince by the ſuppoſition, a is to b as c is to d; it follows by permutation, that a is to c as b is to d; and by compoſition, that $\overline{a+c}$ is to a as $\overline{b+d}$ to b; and again by permutation, that $\overline{a+c}$ is to $\overline{b+d}$ as a is to b: in like manner by permutation and diviſion we ſhall have $\overline{a-c}$ to $\overline{b-d}$ as a to b.

Tenthly, if there be three numbers a, b and c, and other three numbers d, e and f proportionable to them, and in the ſame order, that is, if as a is to b ſo d is to e, and as b is to c ſo e is to f; I ſay then, that *ex aequo*, the extremes will be in the ſame proportion, (*viz.*) that a will be to c as d is to f: for ſince by the ſuppoſition, a is to b as d is to e; by permutation we ſhall have a to d as b to e; and for the ſame reaſon, ſince b is to c as e is to f; we ſhall have b to e as c to f: ſince then a is to

to d as b to e, and b to e as c to f; it follows from the ſecond propoſition, that a is to d as c to f; and by permutation, that a is to c as d to f.

Eleventhly, if there be three numbers, a, b and c, and three other numbers d, e and f proportionable to them, but in a contrary order, ſo that a is to b as e to f, and b to c as d to e; I ſay, that the extremes will ſtill be proportionable, to wit, that a will be to c as d to f: for ſince a is to b as e to f, we have $af = be$; moreover ſince b is to c as d to e, we have $cd = be$; therefore $af = cd$; make a and f extremes, and you will have a to c as d to f.

N. B. If there be two ſerieſes of numbers as a, b, c, *&c*; d, e, f, *&c*, each ſeries conſiſting of the ſame number of terms; and if all the proportions between contiguous terms in one ſeries, be reſpectively equal to all thoſe in the other, that is, each to each, as they ſtand in order; as if a be to b as d to e, and b to c as e to f, *&c*; then the extreme terms of one ſeries will be proportionable to the extreme terms of the other: for the demonſtration of the tenth propoſition may be extended to as many terms as we pleaſe; and this proportionality of the extremes, is ſaid to follow *ex æquo ordinate*, or barely *ex æquo*, that is, from a reſpective equality of all the proportions in one ſeries to their correſpondents in the other, in an orderly manner. But if every proportion in one ſeries, has an equal proportion to anſwer it in the other, but not in a correſpondent part of the ſeries; as if a be to b as e to f, and b to c as d to e, *&c*; then though the extremes will ſtill be proportionable, as will be evident by continuing the demonſtration of this eleventh propoſition; yet now the proportionality of the extremes is ſaid to follow *ex æquo perturbate*, that is, from an equality of all the proportions in one ſeries to all thoſe in the other, but in a diſorderly manner.

Twelfthly, if a is to b as c is to d; we ſhall have $\overline{a+b}$ to $\overline{a-b}$ as $\overline{c+d}$ is to $\overline{c-d}$: for ſince a is to b as c is to d, we ſhall have by compoſition, $\overline{a+b}$ to a as $\overline{c+d}$ is to c; we ſhall have alſo by diviſion, $\overline{a-b}$ to a as $\overline{c-d}$ to c; and by inverſion, a to $\overline{a-b}$ as c to $\overline{c-d}$: ſince then we have $\overline{a+b}$ to a as $\overline{c+d}$ to c; and a to $\overline{a-b}$ as c to $\overline{c-d}$, that is, ſince we have three numbers, $\overline{a+b}$, a, and $\overline{a-b}$, and other three numbers proportionable to them in the ſame order, to wit, $\overline{c+d}$, c, and $\overline{c-d}$; it follows *ex æquo*, that the extremes will be proportionable, that is, that $\overline{a+b}$ will be to $\overline{a-b}$ as $\overline{c+d}$ is to $\overline{c-d}$.

Thirteenthly, if there be a ſeries of numbers, k, l, m, n, whereof k is to l as a to b, and l to m as c to d, and m to n as e to f; I ſay then

that

that k the first term, will be to n the last, as ace the product of all the other antecedents to bdf the product of all the other consequents: for k is to l as a to b, by the supposition; and we shall find that a is to b as ace to bce by multiplying extremes and means; therefore k is to l as ace to bce; and for a like reason l is to m as bce to bde, and m is to n as bde to bdf; therefore *ex æquo*, k is to n as ace to bdf.

Of the extraction of the square roots of simple algebraic quantities.

17. The extraction of the square root of simple algebraic quantities is so very easy, that it needs not to be insisted on. Thus the square root of aa is $+$ or $-a$, the square root of $9aa$ is $+$ or $-3a$, and that of $4aabb$ is $+$ or $-2ab$: this is plain from the definition of the square root; for the square root of any quantity, suppose of $4aabb$, is that, which being multiplied into itself, will produce $4aabb$: now $-2ab$ multiplied into itself will produce $4aabb$, as well as $+2ab$, and therefore one quantity is as much it's square root as the other.

When the square root of a quantity cannot be extracted, it is usual to signific it by this mark $\surd$: thus $\surd 2aa$ signifies the square root of $2aa$; thus $\surd\overline{aa-4b}$ signifies the square root of the whole quantity $\overline{aa-4b}$; thus $\frac{\surd\overline{aa-4b}}{2a}$ signifies a fraction whose numerator is the square root of the whole quantity $\overline{aa-4b}$, and whose denominator is $2a$; thus $\surd\frac{4ab-a^3}{12a}$ signifies the square root of the whole fraction $\frac{4ab-a^3}{12a}$, that is, the square root of both the numerator and denominator.

When the square root of a quantity cannot be extracted, the quantity may sometimes however be resolved into two factors, whereof the one is a square, and the other is not; and whenever this is possible, the root of the square may be extracted, and the radical sign may be prefixed to the other factor: thus $12aa$ equals $4aa \times 3$; therefore $\surd 12aa = 2a \times \surd 3$.

Of the extraction of the square roots of compound algebraic quantities.

18. The extraction of the square root of compound algebraic quantities is so very like that of whole numbers in common Arithmetick, especially in the case of serieses where it is chiefly required, that one would be apt to imagine, a bare inspection of the work would be sufficient to shew the process to any one tolerably well skilled in the latter; but if that be not thought sufficient, take the following directions, with an example.

Let it be required to extract the ſquare root of this quantity $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$: here, pointing every other place from the place of units, that is, from 16, I demand the ſquare root of the member belonging to the firſt point to the left hand, that is, of x^6, and the anſwer is x^3, which I put down in the quotient; then I ſquare x^3, and ſubtract the product x^6, from x^6, the member belonging to the firſt point, and there remains nothing: then I bring down the two next members belonging to the next point, to wit, $4x^5+10x^4$ for a reſolvend, and divide it by $2x^3$, the double of the root in the quotient, that is, I divide $4x^5$ the firſt member of the reſolvend, by $2x^3$, and the quotient is $2x^2$, which I put down in the quotient, as likewiſe after the diviſor $2x^3$; then I multiply the quantity $2x^3+2x^2$ by it's laſt member $2x^2$, and the product $4x^5+4x^4$ I ſubtract from the reſolvend $4x^5+10x^4$, and there remains $6x^4$; to this remainder I bring down the next two places $20x^3+25x^2$, and ſo have a new reſolvend, $6x^4+20x^3+25x^2$; this reſolvend I divide by the double of the root already found, that is, by $2x^3+4x^2$, dividing the firſt member of the reſolvend by the firſt member of this double root, and the quotient is $+3x$, which I put down in the quotient, and alſo after the diviſor $2x^3+4x^2$; then multiplying this laſt quantity $2x^3+4x^2+3x$ by it's laſt member $3x$, the product is $6x^4+12x^3+9x^2$, which being ſubtracted from the laſt reſolvend, leaves $8x^3+16x^2$; to this remainder I bring down the two laſt places $24x+16$, and ſo have a new reſolvend $8x^3+16x^2+24x+16$; this I divide by the double of the root already found, that is, by $2x^3+4x^2+6x$, and the quotient is $+4$; this being put down in the quotient, and alſo after the diviſor $2x^3+4x^2+6x$ gives $2x^3+4x^2+6x+4$; which being multiplied by it's laſt member 4, gives $8x^3+16x^2+24x+16$, which being ſubtracted from the laſt reſolvend, leaves no remainder; therefore the whole root is x^3+2x^2+3x+4. See the work:

$$\begin{array}{rl}
 & \dot{x}^6+4x^5+\dot{1}0x^4+20x^3+\dot{2}5x^2+24x+\dot{1}6\ (x^3+2x^2+3x+4 \\
 & x^6 \\
 & \overline{} \\
2x^3+2x^2 \\ +2x^2 & \left.\begin{array}{l} *+4x^5+10x^4 \\ 4x^5\ +4x^4 \end{array}\right. \\
2x^3+4x^2+3x \\ +3x & \left.\begin{array}{l} *\ +6x^4+20x^3+25x^2 \\ 6x^4+12x^3\ +9x^2 \end{array}\right. \\
2x^3+4x^2+6x+4 \\ +4 & \left.\begin{array}{l} *\ +\ 8x^3+16x^2+24x+16 \\ 8x^3+16x^2+24x+16 \end{array}\right. \\
 & \overline{\ *\quad *\quad *\quad *\ }
\end{array}$$

Other-

Otherwise thus:

$$\begin{array}{lllll}
 & \dot{x}^6+4x^5+\dot{1}0x^4+20x^3+2\dot{5}x^2+24x+\dot{1}6\,(x^3+2x^2+3x+4 & & & \\
 & x^6 & & & \\ \hline
2x^3+2x^2\,\big)\ * & & & & \\
\quad +2x^2\,\big) & +4x^5 & +4x^4 & & \\ \hline
2x^3+4x^2+3x\,\big) & * & +6x^4 & & \\
\quad\quad +3x\,\big) & & 6x^4+12x^3+9x^2 & & \\ \hline
2x^3+4x^2+6x+4\,\big) & & * & +8x^3+16x^2 & \\
\quad\quad\quad +4\,\big) & & & 8x^3+16x^2+24x+16 & \\ \hline
 & & & * \quad * \quad * \quad * &
\end{array}$$

Since the last member $+16$ of the quantity given, was a square, I might have begun the extraction at that end, and the root would have come out the same as before, but inverted.

N. B. By a square number I mean a number that admits of a square root; therefore -16 is no square number, since there is no root either affirmative or negative, which multiplied into itself will produce -16.

For another example, let it be proposed to extract the square root of the following quantity, or at least to reduce the root to the simplest expression, $12x^3-72x^2+108x$: this quantity, because neither of the extremes are squares, must be reduced to a more convenient form thus; if $12x^3$ be divided by $12x$, the quotient xx will be a square number; therefore I divide the whole by $12x$, and the quotient is $xx-6x+9$; whence I conclude *e converso*, that if $xx-6x+9$ be multiplied into $12x$, the product will be the quantity proposed, to wit, $12x^3-72x^2+108x$; therefore by the 8th article, $\sqrt{12x^3-72x^2+108x}$ equals $\sqrt{xx-6x+9}$ into $\sqrt{12x}$: but $\sqrt{x^2-6x+9}$ being extracted as above, is $x-3$, or $3-x$; and because $12x=4\times3x$, we have $\sqrt{12x}=2\times\sqrt{3x}$; therefore, $\sqrt{12x^3-72x^2+108x}=\overline{x-3}$ or $\overline{3-x}$ into $2\times\sqrt{3x}=\overline{2x-6}$ or $\overline{6-2x}$ into $\sqrt{3x}$. If the square root of the factor $xx-6x+9$ could not otherwise have been extracted, it might however have been obtained by an approximation, as is that of a binomial in the following article.

To extract the ſquare root of a binomial by an infinite ſeries.

19. By a binomial in this place I mean any quantity conſiſting of two ſimple quantities connected together by the ſign + or —, as $\overline{a+b}$, $\overline{1-x}$, &c: now though the ſquare root of ſuch a quantity cannot be expreſſed by any finite number of terms, yet by an infinite ſeries it may, as in the following caſes.

Caſe the 1ſt. The ſquare root of the binomial $1+x$, ſuppoſing x to be leſs than one, is found by the following operation to be $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+$ &c; a ſeries, whoſe regularity is not immediately perceived, but I ſhall make it appear in the courſe of this work, that theſe ſerieſes are as regular as any others whatever, and as eaſily computed.

$$
\begin{array}{lll}
 & 1+x \quad * \quad * \quad * \quad * \quad * \quad * \quad * & \left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\&c.\right. \\
 & 1 & \\
2+\frac{x}{2}\Big) & * \quad +x & \\
+\frac{x}{2} & x+\frac{x^2}{4} & \\
2+x-\frac{x^2}{8}\Big) & * \quad -\frac{x^2}{4} & \\
-\frac{x^2}{8} & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} & \\
2+x-\frac{x^2}{4}+\frac{x^3}{16}\Big) & * \quad +\frac{x^3}{8}-\frac{x^4}{64} & \\
+\frac{x^3}{16} & \frac{x^3}{8}+\frac{x^4}{16} & \\
2+x-\frac{x^2}{4}+\frac{x^3}{8}-\frac{5x^4}{128}\Big) & * \quad -\frac{5x^4}{64} & \\
-\frac{5x^4}{128} & -\frac{5x^4}{64} & \\
 & * &
\end{array}
$$

Since

Since in all these serieses, the powers of x ascend or descend regularly, these roots may be extracted by the help of the coefficients only, and the powers may be supplied afterwards; but then due care must be taken to keep the terms distinct, so as to place all of the same kind orderly one under another, that no confusion may arise in the operation.

Let us now try the root thus found, and see what it will produce when multiplied into itself: but first I must advertise the reader, that as this root is not extended beyond the fourth power of x, so neither ought it to be expected that the square of this root when multiplied into itself, should be true to more places than the root; therefore in this multiplication, all the powers of x beyond the fourth ought to be excluded out of the product as useless terms. Now in order to square the quantity $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$, it must be multiplied,

First into 1, and the product will be	$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$ &c.
then into $\frac{1}{2}x$, and the product will be	$+\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{16}x^3+\frac{1}{32}x^4$ &c.
then into $-\frac{1}{8}x^2$, and the product will be	$-\frac{1}{8}x^2-\frac{1}{16}x^3+\frac{1}{64}x^4$ &c.
then into $\frac{1}{16}x^3$, and the product will be	$+\frac{1}{16}x^3+\frac{1}{32}x^4$ &c.
then into $-\frac{5}{128}x^4$, and the product will be	$-\frac{5}{128}x^4$ &c.
add these together, and the sum will be as it ought.	$1+x$ * * *

As in squaring this root, no notice was taken of any powers of x beyond the fourth, so likewise in extracting the square root, the work must not be suffered to run on beyond the power where the series is designed to terminate, that is, all other powers must be excluded out of the work, as they are out of the root.

Case 2d. Let now the square root of the binomial $1-z$ be required: where it is plain that z must be less than 1, otherwise $1-z$ would be a negative quantity, and could have no square root: put then $-z$ instead of x in the foregoing case, and you will have $xx=+z^2$, $x^3=-z^3$, $x^4=+z^4$ &c; $\frac{1}{2}x=-\frac{1}{2}z$, $-\frac{1}{8}x^2=-\frac{1}{8}z^2$, $+\frac{1}{16}x^3=-\frac{1}{16}z^3$, $-\frac{5}{128}x^4=-\frac{5}{128}z^4$ &c: therefore $\sqrt{1-z}=1-\frac{1}{2}z-\frac{1}{8}z^2-\frac{1}{16}z^3-\frac{5}{128}z^4-$&c.

Case 3d. Let the square root of the binomial $z+1$ be required, supposing z to be greater than 1: here, though 1 be a square number, yet as this root is to be expressed by an infinite series perpetually converging nearer and nearer to it; the series must take it's rise from the greater part of the binomial, that is, from z: divide then the quantity $z+1$ by z, and

and the quotient will be $1+\frac{1}{z}$, where $\frac{1}{z}$ will be leſs than 1, becauſe z is greater: ſince then $\frac{z+1}{z}$ equals $1+\frac{1}{z}$, it follows *e converſo*, that $\overline{1+\frac{1}{z}}\times z=z+1$; therefore $\sqrt{\overline{1+\frac{1}{z}}}\times\sqrt{z}=\sqrt{\overline{z+1}}$: put $\frac{1}{z}$ inſtead of x in the firſt caſe, and you will have $\sqrt{\overline{1+\frac{1}{z}}}=1+\frac{1}{2}\times\frac{1}{z}-\frac{1}{8}\times\frac{1}{z^2}+\frac{1}{16}\times\frac{1}{z^3}-\frac{5}{128}\times\frac{1}{z^4}$ &c. $=1+\frac{1}{2z}-\frac{1}{8z^2}+\frac{1}{16z^3}-\frac{5}{128z^4}$ &c: call the ſum of this ſeries continued *ad infinitum* s, and you will have $\sqrt{\overline{z+1}}=s\sqrt{z}$, that is, $s\times\sqrt{z}$.

Caſe 4th. Let it be required to expreſs by an infinite ſeries, the ſquare root of the binomial $a+b$, ſuppoſing a to be the greater part of the binomial: here $\frac{a+b}{a}=1+\frac{b}{a}$; therefore $\overline{1+\frac{b}{a}}\times a=a+b$; but $\sqrt{\overline{1+\frac{b}{a}}}$ by the firſt caſe is $1+\frac{b}{2a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}$ &c: call the ſum of this ſeries continued *ad infinitum* s, and you will have $\sqrt{\overline{a+b}}=s\sqrt{a}$.

Preparations for demonſtrating the rule for finding the greateſt common meaſure of two numbers given; that is, their greateſt common diviſor.

20. Coming now to give ſome examples of the ſeveral reductions and operations of algebraic fractions; I ſhall take this occaſion before I begin, to give ſome account of the rule for finding the greateſt common meaſure to any two numbers propoſed; a rule very uſeful in vulgar fractions, and ſometimes in Algebra, when both the numerator and denominator of a fraction are compound quantities.

This ſhould have been done ſooner, but that I made ſome doubt whether ſuch a demonſtration of this rule could be given, as might be ſuited to the capacity of a young beginner; nay I am not abſolutely certain whether even ſtill he will be able to comprehend it: but if he finds it too difficult for him, the beſt advice I can give him is, to paſs it by till he finds his imagination more confirmed, his attention more fixed, and his reaſon improved by habit and exerciſe.

For

For the better underſtanding of this rule and the demonſtration belonging to it, I ſhall lay down the following definitions and axioms.

Def. 1. *One number is ſaid to meaſure another, when it is exactly contained in that other without any overplus or remainder.* Thus 3 may be ſaid to meaſure 12, as being exactly four times contained in it.

2. *A number is ſaid to be a common meaſure to two others, when it meaſures them both.* Thus 3 is a common meaſure both to 12 and 21.

Axiom 1ſt.

If one quantity meaſures another, and that other meaſures a third, then will the firſt meaſure the third. As if 3 meaſures 12, and 12 meaſures 24, 3 will meaſure 24.

Axiom 2d.

If a number be a common meaſure to two others, it will meaſure both their ſum and their difference. Thus 3, which is a common meaſure to 12 and 21, will meaſure both their ſum 33, and their difference 9; for if 3 be four times contained in 12, and ſeven times in 21, it ought to be eleven times contained in their ſum, becauſe $4+7=11$; and it ought to be three times contained in their difference, becauſe $7-4=3$.

Axiom 3d.

If one number be divided by another, and there be any remainder, take away the remainder from the dividend, and the diviſor will meaſure the reſt. Thus if 14 be divided by 3, and there remains 2, take away 2 from 14, and 3 will meaſure the reſt, which is 12. In general thus: *If* a *be divided by* b, *and there remains* c, *then will* b *meaſure* $\overline{a-c}$.

The rule explained and demonſtrated.

21. Theſe things being laid down, I come now to the rule for finding the greateſt common meaſure to any two numbers propoſed, which is as follows: *Let* a *and* b *be the two numbers whoſe greateſt common meaſure is wanted, to wit,* a *the greater, and* b *the leſs: let* a *be divided by* b, *and without any regard had to the quotient, let* c *be the remainder; then divide* b *by* c, *and let the remainder be* d; *then divide* c *by* d, *and let the remainder be* e; *laſtly divide* d *by* e, *and let there be no remainder at all; I ſay then, that this laſt diviſor* e, *which had no remainder, will not only be a common meaſure to* a *and* b, *but will be the greateſt common meaſure they will admit of.*

But firſt I ſhall prove, that e is a common meaſure to a and b, which is thus done:

1ſt. e meaſures d *ex hypotheſi*; and d meaſures $\overline{c-e}$, by the third axiom, becauſe when c was divided by d, the remainder was e; therefore e meaſures $\overline{c-e}$, by the firſt axiom; but e meaſures itſelf; therefore e meaſures both e and $\overline{c-e}$; but the ſum of e and $\overline{c-e}$ is c; therefore e meaſures c by the ſecond axiom.

2dly. It has now been proved that e meaſures c; but c meaſures $\overline{b-d}$; therefore e meaſures $\overline{b-d}$; but e meaſures d *ex hypotheſi*; therefore e meaſures both d and $\overline{b-d}$; but the ſum of d and $\overline{b-d}$ is b; therefore e meaſures b.

3dly. It has been proved that e meaſures b; and b meaſures $\overline{a-c}$; therefore e meaſures $\overline{a-c}$; but it was proved before, that e meaſures c; therefore e meaſures both c and $\overline{a-c}$; but the ſum of c and $\overline{a-c}$ is a; therefore e meaſures a. Thus then we have proved the firſt part of our aſſertion, which was, that e meaſures both a and b.

We ſhall in the next place, by tracing back the ſteps of the former demonſtration, prove, that e is the greateſt quantity that will meaſure them both: for, if poſſible, let us ſuppoſe f to be greater than e, and yet to be a common meaſure to a and b, and let us ſee what will be the conſequence of this ſuppoſition.

1ſt. f meaſures b, and b meaſures $\overline{a-c}$; therefore f meaſures $\overline{a-c}$; but f meaſures a by the ſuppoſition; therefore f meaſures both a and $\overline{a-c}$; but the difference between a and $\overline{a-c}$ is c; therefore f meaſures c, by the ſecond axiom.

2dly. f meaſures c, as above, and c meaſures $\overline{b-d}$; therefore f meaſures $\overline{b-d}$; but f meaſures b by the ſuppoſition; therefore f meaſures both b and $\overline{b-d}$; but the difference between b and $\overline{b-d}$ is d; therefore f meaſures d.

3dly. Again, f meaſures d as above, and d meaſures $\overline{c-e}$; therefore f meaſures $\overline{c-e}$; but f meaſures c, as was proved before; therefore f meaſures both c and $\overline{c-e}$; but the difference between c and $\overline{c-e}$ is e; therefore f meaſures e, that is, a greater quantity meaſures a leſs, which is abſurd; therefore the ſuppoſition that a and b would admit of a greater common meaſure than e, from whence this abſurdity flowed, was falſe; for truth never leads to abſurdities: therefore e is the greateſt common meaſure the two numbers a and b will admit of. *Q. E. D.*

N. B.

N. B. This demonstration is *Euclid*'s a little changed, and may serve as a specimen of the subtilty of the Ancients.

For an example to the foregoing rule, see the introduction, art. 7th.

COROLLARIES.

1st. If in finding the greatest common measure we can have no divisor without any remainder till we come to unity, we ought then to conclude, that the numbers proposed are *primi inter se*, that is, are such as will admit of no common measure but unity: for *As every number is called a prime number, that will admit of no divisor but itself and unity; so two numbers are said to be prime to each other, that will admit of no common divisor but unity.*

2d. *Whatever number will measure two others, it will also measure their greatest common measure:* thus upon the supposition that f would measure both a and b, it was proved to measure e, their greatest common measure.

The several rules of fractions exemplified in Algebraic quantities.

22. Fractions in Algebra are treated just in the same manner as in common arithmetic, only using algebraical instead of numeral operations; as will plainly appear from the following examples.

Examples of the reduction of fractions from higher to lower terms, according to introduction art. 7th.

The fraction $\frac{4ab}{6bc}$, dividing both the numerator and denominator by the same quantity $2b$, will be reduced to the fraction $\frac{2a}{3c}$, a fraction of the same value with the former, but expressed in more simple terms: whence we may infer, that whenever a common letter or factor is to be found in every member both of the numerator and denominator, it may be cancelled every where, without affecting the value of the fraction: thus the fraction $\frac{ac+bc}{cd+ce}$, expunging c, becomes $\frac{a+b}{d+e}$, a fraction of the same value. But if there be any one member, wherein the factor is not concerned, it must not be expunged at all: thus the fraction $\frac{ac+bc}{cd+e}$ cannot be reduced, because the factor c is not to be found in e.

Note, That cancelling here, is not subtracting, but dividing: thus to cancel the letter b in the quantity ab, so as to reduce it to a, is not to

M subtract

ſubtract b from ab, but to divide ab by b, in which caſe the quotient will be a.

Examples of fractions reduced to the ſame denomination, according to introduction art. 8th.

1ſt. The fractions $\frac{a}{2}$, $\frac{b}{3}$ and $\frac{c}{4}$, when reduced to the ſame denomination, will ſtand thus; $\frac{12a}{24}$, $\frac{8b}{24}$ and $\frac{6c}{24}$. *2d.* The fractions $\frac{a}{b}$ and $\frac{c}{d}$ ſo reduced, will ſtand thus; $\frac{ad}{bd}$ and $\frac{bc}{bd}$. *3d.* The fractions $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{u}$ and $\frac{x}{y}$, after reduction, will ſtand thus; $\frac{psuy}{qsuy}$, $\frac{qruy}{qsuy}$, $\frac{qsty}{qsuy}$ and $\frac{qsux}{qsuy}$. And here I cannot but obſerve, that now the rule for this reduction demonſtrates itſelf: for in this example it is impoſſible not to ſee, that all theſe fractions, notwithſtanding this reduction, ſtill retain their former values: thus the firſt fraction $\frac{psuy}{qsuy}$, by cancelling common factors, is reduced to $\frac{p}{q}$, it's former value; and the ſame may be obſerved of all the reſt: and this example amounts to a demonſtration, becauſe it is comprehended in general terms. But to go on: *4th.* The fractions $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ being reduced to the ſame denomination, become $\frac{bc}{abc}$, $\frac{ac}{abc}$ and $\frac{ab}{abc}$. *5th.* And laſtly, $\frac{1}{a+b}$ and $\frac{1}{a-b}$ when thus reduced, become $\frac{a-b}{aa-bb}$ and $\frac{a+b}{aa-bb}$: for 1 the numerator of the firſt fraction multiplied into $a-b$, the denominator of the ſecond, makes $a-b$; and 1 the numerator of the ſecond fraction multiplied into $a+b$, the denominator of the firſt, makes $a+b$; and the product of the two denominators $a+b$ and $a-b$ multiplied together is $aa-bb$, as in the 4th example of the 9th article.

Examples of addition in fractions, according to introduction art. 9th.

1ſt. Theſe fractions $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{-c}{2}$, when added together, make $\frac{a+b-c}{2}$.

2d. The fraction $\frac{a+b}{2}$ added to the fraction $\frac{a-b}{2}$ makes $\frac{2a}{2}$ or a.

3d. The fractions $\frac{a}{2}$, $\frac{-b}{3}$, and $\frac{+c}{4}$ when added together, make $\frac{12a-8b+6c}{24}$.

4th. The fraction $\frac{a}{b}$ added to the fraction $\frac{c}{d}$ makes $\frac{ad+bc}{bd}$.

5th. a added to $\frac{b}{c}$, that is, $\frac{a}{1}$ added to $\frac{b}{c}$ makes $\frac{ac+b}{c}$,

6th. $\frac{1}{a}$ added to $-\frac{1}{b}$ makes $\frac{b-a}{ab}$.

7th. The fractions $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{u}$ and $\frac{x}{y}$, when added together, make $\frac{psuy+qruy+qsty+qsux}{qsuy}$.

8th. $\frac{a}{b}$ added to $\frac{1}{c}$ gives $\frac{ac+b}{bc}$.

9th. $\frac{1}{a+b}$ added to $\frac{1}{a-b}$ gives $\frac{2a}{aa-bb}$. See the 5th example of fractions reduced to the same denomination.

Examples of subtraction in fractions, according to introduction art. 10th.

Note first, If the signs of both the numerator and denominator of any fraction be changed, which is no more than multiplying both terms into -1, the value of the fraction will still remain.

Secondly, The denominator of a fraction is always supposed to be affirmative; and therefore if at any time it happens to be otherwise, it must be made affirmative by changing the signs of both terms.

Thirdly, $+\frac{a}{b}$ and $-\frac{a}{b}$ are the same in effect as $\frac{+a}{b}$ and $\frac{-a}{b}$, as is evident from the nature of division: and sometimes, this latter way of notation is more convenient than the former.

Fourthly, Therefore the sign of the numerator is the sign of the whole fraction; and to change the sign of the former, is the same in effect, as to change the sign of the latter.

Fifthly, Whenever one algebraic fraction is to be ſubtracted from another, the ſafeſt way will be to change the ſign of the numerator of the fraction to be ſubtracted, and to place it after the other, and then to reduce them at laſt into one fraction: for if the ſubtraction be deferred till after the reduction is over, one may make a miſtake, and ſubtract the wrong quantity. Thus 1ſt. $\frac{4b}{5}$ ſubtracted from $\frac{2a}{3}$ gives $\frac{2a}{3} \frac{-4b}{5} = \frac{10a-12b}{15}$.

2d. $\frac{r}{s}$ ſubtracted from $\frac{p}{q}$ gives $\frac{p}{q} \frac{-r}{s} = \frac{ps-qr}{qs}$.

3d. $\frac{b}{c}$ ſubtracted from a, gives $\frac{a}{1} \frac{-b}{c} = \frac{ac-b}{c}$.

4th. $\frac{1}{a+b}$ ſubtracted from $\frac{1}{a-b}$, gives $\frac{1}{a-b} \frac{-1}{a+b} = \frac{2b}{aa-bb}$.

Examples of multiplication in fractions.

The multiplication of fractions is performed, by multiplying the numerator and denominator of the multiplicand, into the numerator and denominator of the multiplicator reſpectively.

Thus 1ſt. $\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}$.

2d. $\frac{3p}{4q} \times \frac{5q}{6r} = \frac{15pq}{24qr} = \frac{5p}{8r}$.

3d. $\frac{a}{b} \times c$ or $\frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}$.

4th. $\frac{a}{b} \times b = \frac{ab}{b} = a$.

5th. $\frac{3a}{4b} \times 20b = \frac{60ab}{4b} = 15a$.

6th. $\frac{4a}{5} \times \frac{7}{8a} = \frac{28a}{40a} = \frac{7}{10}$.

7th. $\frac{3a}{4b} \times \frac{3a}{4b} = \frac{9aa}{16bb}$.

8th.

8th. $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ace}{bdf}$

9th. $\overline{a+\frac{b}{c}}\times d$, or $\frac{\overline{ac+b}}{c}\times\frac{d}{1}=\frac{acd+bd}{c}$.

10th. $\overline{d+\frac{e}{f}}\times\frac{g}{h}$, or $\frac{df+e}{f}\times\frac{g}{h}=\frac{dfg+eg}{fh}$.

11th. $\overline{a+\frac{b}{c}}\times\overline{d+\frac{e}{f}}$, or $\frac{ac+b}{c}\times\frac{df+e}{f}=\frac{acdf+ace+bdf+be}{cf}$.

This multiplication might also be performed thus:

$$\begin{array}{l} d+\frac{e}{f} \\ a+\frac{b}{c} \\ \hline ad+\frac{ae}{f}+\frac{bd}{c}+\frac{be}{cf}. \end{array}$$

12th. $\overline{a+\frac{b}{c}}\times\overline{a+\frac{b}{c}}=\frac{aacc+2abc+bb}{cc}$

Or $aa+\frac{2ab}{c}+\frac{bb}{cc}$. See the work:

$$\begin{array}{l} a+\frac{b}{c} \\ a+\frac{b}{c} \\ \hline aa+\frac{ab}{c}+\frac{bb}{cc} \\ \quad +\frac{ab}{c} \\ \hline aa+\frac{2ab}{c}+\frac{bb}{cc}. \end{array}$$

Examples of diviſion in fractions.

Diviſion in fractions is performed by multiplying the direct terms of the dividend into the inverted terms of the diviſor: thus,

1ſt. $\frac{r}{s}) \; \frac{p}{q} \; (\frac{ps}{qr}$. 2d. $\frac{b}{c}) \; \frac{1}{a} \; (\frac{c}{ab}$.

3d. $\frac{1}{c}) \; \frac{a}{b} \; (\frac{ac}{b}$. 4th. $c) \; \frac{a}{b} \; (\frac{a}{bc}$.

5th. $\frac{1}{c}) \; \frac{ab}{c} \; (\frac{abc}{c}$, or ab. 6th. $\overline{d+\frac{e}{f}}) \; a+\frac{b}{c} \; (\frac{acf+bf}{cdf+ce}$.

7th. $\sqrt{b}) \; \sqrt{a} \; (\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$: for if we make $x = \frac{\sqrt{a}}{\sqrt{b}}$ we ſhall have $xx = \frac{a}{b}$, and $x = \sqrt{\frac{a}{b}}$.

I ſhall only give one example more, and that ſhall be of the rule of proportion, as follows: *If* $\frac{a}{b}$ *gives* $\frac{c}{d}$ *what will* $\frac{e}{f}$ *give?* Anſwer, $\frac{bce}{adf}$: for $\frac{c}{d}$ the ſecond number multiplied into $\frac{e}{f}$ the third, produces $\frac{ce}{df}$; and this divided by the firſt $\frac{a}{b}$ quotes $\frac{bce}{adf}$.

Of equations in Algebra; and particularly of ſimple equations, together with the manner of reſolving them.

23. An equation in Algebra is a propoſition wherein one quantity is declared equal to another, or where one expreſſion of any quantity is declared equal to another expreſſion of the ſame quantity: as when we ſay $\frac{3}{4} = \frac{6}{8}$; where $\frac{3}{4}$ is ſaid to poſſeſs one ſide of the equation, and $\frac{6}{8}$ the other.

An affected quadratic equation is an equation conſiſting of three different ſorts of quantities; one wherein the ſquare of the unknown quantity is concerned, another wherein the unknown quantity is ſimply concerned, and a third wherein it is not concerned at all: as if $xx - 2x = 3$; ſuppoſing x to be an unknown quantity.

If either the term wherein the ſimple power of x is concerned, as $-2x$, or that which is called the abſolute term, to wit, 3, be wanting, the equation

tion is ſtill a quadratic equation, though incompleat. Some indeed there are, who rank this latter ſort of equations under the denomination of ſimple equations; and ſo ſhall we, upon account of their eaſy reſolution; though properly ſpeaking, a ſimple equation is that wherein ſome ſimple power of the unknown quantity is concerned, all others being excluded: as if $3x=6$; $2x+3=4x-5$, &c.

The uſe of theſe equations is for repreſenting more conveniently and more diſtinctly the conditions of problems, when tranſlated out of common language into that of Algebra. As for example; let it be propoſed to find a number with the following property, to wit, that $\frac{2}{3}$ of it with 4 over, may amount to the ſame as $\frac{7}{12}$ of it with 9 over: here, putting x for the unknown quantity, the condition of this problem, when tranſlated out of common language into that of Algebra, will be repreſented by the following equation, to wit, $\frac{2x}{3}+4=\frac{7x}{12}+9$: for $\frac{2}{3}$ of x, that is, $\frac{2}{3}$ of $\frac{x}{1}$ is $\frac{2x}{3}$; therefore $\frac{2x}{3}+4$ ſignifies $\frac{2}{3}$ of x with 4 over; and ſince this expreſſion according to the problem, amounts to the ſame with the other, to wit, $\frac{7x}{12}+9$; hence it is that we pronounce them equal to one another.

Now ſince in the foregoing equation, as well as in almoſt all others ariſing immediately from the conditions of problems themſelves, the unknown quantity is embarraſſed and entangled with ſuch as are known, the way to diſengage it from ſuch known quantities, ſo that itſelf alone poſſeſſing one ſide of the equation, may be found equal to ſuch as are entirely known on the other, that is, in the preſent caſe, to determine the value of the unknown quantity x, is what is commonly called the reſolution of an equation: for the effecting of which, ſeveral axioms and proceſſes are required; ſome whereof, namely ſuch as moſt frequently occur, I ſhall here put down; the reſt I ſhall take notice of occaſionally, as they offer themſelves.

Of the reſolution of ſimple equations.

AXIOM I.

Whenever a fraction is to be multiplied by a whole number, it will be ſufficient to multiply only the numerator by that number, retaining the denominator the ſame as before. Thus $\frac{4}{5}$ multiplied into 2, gives $\frac{8}{5}$, for the ſame reaſon that 4 ſhillings multiplied into 2 gives 8 ſhillings: thus in the firſt example following, $\frac{7x}{12}$ multiplied into 3, gives $\frac{21x}{12}$.

AXIOM 2.

But if the whole number into which the fraction is to be multiplied, be equal to the denominator of the fraction, then throw away the denominator, and the numerator alone will be the product. Thus the fraction $\frac{a}{b}$ multiplied into b, gives $\frac{ab}{b}$ or a: thus in the first example, $\frac{2x}{3}$ multiplied into 3, gives $2x$; and $\frac{21x}{12}$ multiplied into 12 gives $21x$.

AXIOM 3.

If the two sides of an equation be multiplied or divided by the same number, the two products, or quotients, will still be equal to each other. Thus in the first example, where $\frac{2x}{3}+4=\frac{7x}{12}+9$; if both sides of the equation be multiplied into 3, we shall have $2x+12=\frac{21x}{12}+27$; and if again this last equation be multiplied into 12, we shall have $24x+144=21x+324$.

AXIOM 4.

If a quantity be taken from either side of an equation, and placed on the other with a contrary sign, which is commonly called transposition, the two sides will still be equal to each other. Thus if $7+3=10$, transpose $+3$, and you will have $7=10-3$: thus if $7-3=4$, transpose -3, and you will have $7=4+3$: thus if (as in the first example) $24x+144=21x+324$, transpose $21x$, and you will have $24x-21x+144=324$, that is, $3x+144=324$; and if again in this last equation you transpose 144, you will have $3x=324-144=180$.

Transposition therefore, as it is here delivered, is nothing but a general name for adding or subtracting equal quantities from the two sides of an equation; in which case it is no wonder, if the sums or differences still continue equal to each other. As for instance, in this equation $a-b=c$, transposing $-b$ we have $a=c+b$: and what is this after all, but adding b to both sides of the equation? for if b be added to $a-b$, the sum will be a; and if b be added to c, the sum will be $c+b$; therefore $a=c+b$: again, in the equation $a+b=c$, transposing $+b$ we have $a=c-b$, which is nothing else but subtracting b from both sides of the equation.

The

The 1st Process.

If, when an equation is to be resolved, fractions be found on one, or both sides, it must be freed from them by multiplying the whole equation into the denominators of those fractions successively.

The 2d Process.

After the equation is thus reduced to integral terms, if the unknown quantity be found on both sides the equation, let it be brought by transposition to one and the same side, viz. *to that side, which after reduction will exhibit it affirmative.*

The 3d Process.

After this, if any loose known quantities be found on the same side with the unknown, let them also be brought by transposition to the other side of the equation.

The 4th Process.

If now the unknown quantity has any coefficient before it, divide all by that coefficient, and the equation will be resolved.

The 5th Process.

If the whole equation can be divided by the unknown quantity, let such a division be made, and the equation will be reduced to a more simple one. Thus in the 16th example you have $615x - 7xxx = 48x$; divide the whole equation by x, and you will have $615 - 7xx = 48$. In the 13th example you have $\frac{42x}{x-2} = \frac{35x}{x-3}$; divide the whole by x, which is done by dividing only the numerators of the two fractions, and you will have $\frac{42}{x-2} = \frac{35}{x-3}$.

The 6th Process.

If at last the square of the unknown quantity, and not the unknown quantity itself, appears to be equal to some known quantity on the other side of the equation, then the unknown quantity must be made equal to the square root of that which is known. Thus in the 14th example we have $xx = 36$; therefore $x = 6$, and not 18: in the 15th, we have $xx = 64$; therefore $x = 8$, the square root of 64, and not 32, it's half.

Examples of the resolution of simple equations.

24. This preparation being made, I shall now give some examples of the resolution of simple equations; and my first example shall be the equation given in the last article, in order to trace out the number there described.

Example 1.

$$\frac{2x}{3}+4=\frac{7x}{12}+9.$$

In this equation it is plain, that there are two fractions, $\frac{2x}{3}$, and $\frac{7x}{12}$, which must be taken off at two several operations, thus: as 3 is the denominator of the first fraction, multiply the whole equation by 3, and you will have $2x+12=\frac{21x}{12}+27$: again, as the denominator of the remaining fraction is 12, multiply all by 12, and you will have $24x+144=21x+324$; which is an equation free from fractions.

2dly, It must in the next place be considered, that in this last equation $24x+144=21x+324$, the unknown quantity is concerned on both sides, to wit, $24x$ on one side, and $21x$ on the other; transpose therefore $21x$, and you will have $24x-21x+144=324$, that is, $3x+144=324$. If it be asked why I chose to transpose $21x$ rather than $24x$; my answer is, that had $24x$ been transposed, the unknown quantity, or it's coefficient at least, after reduction, would have been negative, contrary to the rule in the second process; for resuming the equation $24x+144=21x+324$, if $24x$ be transposed, we shall have $144=21x-24x+324$, that is, $144=-3x+324$: but even in this case, another transposition will set all right; for if $-3x$ be transposed in this last equation, we shall then have $3x+144=324$ as before: all that can be said then against this last way is, that it creates unnecessary transpositions, which an artist would always endeavour to avoid.

3dly, Having now reduced the equation to a much greater degree of simplicity than before, to wit, $3x+144=324$; because the unknown quantity $3x$ has still a loose quantity, *viz.* 144 joined with it, transpose that quantity 144 to the other side of the equation, and you will have $3x=324-144$, that is, $3x=180$.

N. B. By a loose quantity I mean such a one as is joined with the unknown by the sign + or —, and not by way of multiplication, as is the coefficient 3 in the last equation.

4thly, By this time the quantity x is very near being diſcovered; for if $3x=180$, it is but dividing all by 3, and we ſhall have $x=60$: 60 therefore is the number deſcribed in the laſt article by this property, to wit, that $\frac{2}{3}$ of it with 4 over, will amount to the ſame as $\frac{7}{12}$ of it with 9 over: and that 60 has this property, will now be eaſily made to appear ſynthetically; for $\frac{2}{3}$ of 60 is 40, and this with 4 over is 44; moreover $\frac{7}{12}$ of 60 is 35, and this with 9 over is alſo 44.

N. B. A demonſtration that proves the connexion between any number and the property aſcribed to it, is either analytical or ſynthetical: if this connexion is ſhewn by tracing the number from the property, the demonſtration of it is called an analytical demonſtration; but if it is ſhewn by tracing the property from the number, the demonſtration is then ſaid to be ſynthetical.

Example 2.

$$\frac{2x}{3}+12=\frac{4x}{5}+6.$$

Here multiply by 3, and you will have $2x+36=\frac{12x}{5}+18$; multiply again by 5, and you will have $10x+180=12x+90$; tranſpoſe $10x$, and you will have $180=12x-10x+90$, that is, $180=2x+90$, or rather $2x+90=180$, for I generally chooſe to have the unknown quantity on the firſt ſide of the equation: tranſpoſe 90, and you will have $2x=180-90$, that is, $2x=90$; divide by 2, and you will have $x=45$.

The Proof.

The original equation was $\frac{2x}{3}+12=\frac{4x}{5}+6$: now if $x=45$, we have $\frac{2x}{3}=30$, and $\frac{2x}{3}+12=42$: again, we have $\frac{4x}{5}=36$, and $\frac{4x}{5}+6=42$; therefore $\frac{2x}{3}+12=\frac{4x}{5}+6$, becauſe the amount of both is 42.

Example 3.

$\frac{3x}{4}+5=\frac{5x}{6}+2$: therefore $3x+20=\frac{20x}{6}+8$; therefore $18x+120=20x+48$; therefore $120=20x-18x+48$, that is, $120=2x+48$; therefore $120-48=2x$, that is, $2x=72$; therefore $x=36$.

The Proof.

The original equation was $\frac{3x}{4}+5=\frac{5x}{6}+2$: now if $x=36$, we shall have $\frac{3x}{4}=27$, and $\frac{3x}{4}+5=32$: we shall also have $\frac{5x}{6}=30$, and $\frac{5x}{6}+2=32$; therefore if $x=36$, we shall have $\frac{3x}{4}+5=\frac{5x}{6}+2$.

Example 4.

$\frac{7x}{8}-5=\frac{9x}{10}-8$: therefore $7x-40=\frac{72x}{10}-64$; therefore $70x-400=72x-640$; therefore $-400=72x-70x-640$, that is, $-400=2x-640$, or rather $2x-640=-400$; therefore $2x=640-400$, that is, $2x=240$; and $x=120$.

The Proof.

The original equation, $\frac{7x}{8}-5=\frac{9x}{10}-8$; but $x=120$; therefore $\frac{7x}{8}=105$; therefore $\frac{7x}{8}-5=100$: moreover $\frac{9x}{10}=108$; therefore $\frac{9x}{10}-8=100$; therefore $\frac{7x}{8}-5=\frac{9x}{10}-8$.

Example 5.

$\frac{5x}{9}-8=74-\frac{7x}{12}$: therefore $5x-72=666-\frac{63x}{12}$; therefore $60x-864=7992-63x$; therefore $60x+63x-864=7992$, that is, $123x-864=7992$; therefore $123x=7992+864$, that is, $123x=8856$; and $x=72$.

The Proof.

The original equation, $\frac{5x}{9}-8=74-\frac{7x}{12}$; $x=72$; therefore $\frac{5x}{9}=40$; therefore $\frac{5x}{9}-8=32$: again, $\frac{7x}{12}=42$; therefore $74-\frac{7x}{12}=74-42=32$; therefore $\frac{5x}{9}-8=74-\frac{7x}{12}$.

Example

Example 6.

$\frac{x}{6}-4=24-\frac{x}{8}$: therefore $x-24=144-\frac{6x}{8}$; therefore $8x-192=1152-6x$; therefore $8x+6x-192=1152$, that is, $14x-192=1152$; therefore $14x=1152+192$, that is, $14x=1344$; and $x=96$.

The Proof.

The original equation, $\frac{x}{6}-4=24-\frac{x}{8}$; $x=96$; $\frac{x}{6}=16$; $\frac{x}{6}-4=12$: again, $\frac{x}{8}=12$; therefore $24-\frac{x}{8}=24-12=12$; therefore $\frac{x}{6}-4=24-\frac{x}{8}$.

Example 7.

$56-\frac{3x}{4}=48-\frac{5x}{8}$: therefore $224-3x=192-\frac{20x}{8}$; therefore $1792-24x=1536-20x$; therefore $1792=1536+24x-20x$, that is, $1792=1536+4x$; therefore $1792-1536=4x$, that is, $4x=256$; and $x=64$.

The Proof.

The original equation, $56-\frac{3x}{4}=48-\frac{5x}{8}$; $x=64$; therefore $\frac{3x}{4}=48$; therefore $56-\frac{3x}{4}=56-48=8$: again, $\frac{5x}{8}=40$; therefore $48-\frac{5x}{8}=48-40=8$; therefore $56-\frac{3x}{4}=48-\frac{5x}{8}$.

Example 8.

$36-\frac{4x}{9}=8$: therefore $324-4x=72$; therefore $324=72+4x$; therefore $324-72=4x$, that is, $4x=252$; and $x=63$.

The Proof.

The original equation, $36-\frac{4x}{9}=8$; $x=63$; therefore $\frac{4x}{9}=28$; therefore $36-\frac{4x}{9}=36-28=8$.

Example

Example 9.

$\frac{2x}{3}=\frac{176-4x}{5}$: therefore $2x=\frac{528-12x}{5}$; therefore $10x=528-12x$; therefore $10x+12x=528$, that is, $22x=528$; and $x=24$.

The Proof.

The original equation, $\frac{2x}{3}=\frac{176-4x}{5}$; $x=24$; therefore $\frac{2x}{3}=16$: again, $4x=96$; therefore $176-4x=176-96=80$; therefore $\frac{176-4x}{5}=\frac{80}{5}=16$; therefore $\frac{2x}{3}=\frac{176-4x}{5}$.

Example 10.

$\frac{3x}{4}+\frac{180-5x}{6}=29$: therefore $3x+\frac{720-20x}{6}=116$; therefore $18x+720-20x=696$, that is, $720-2x=696$; therefore $720=2x+696$; therefore $720-696=2x$, that is, $2x=24$; and $x=12$.

The Proof.

The original equation, $\frac{3x}{4}+\frac{180-5x}{6}=29$; $x=12$; therefore $\frac{3x}{4}=9$; $5x=60$; therefore $180-5x=180-60=120$; therefore $\frac{180-5x}{6}=\frac{120}{6}=20$; therefore $\frac{3x}{4}+\frac{180-5x}{6}=29$.

Example 11.

$$\frac{45}{2x+3}=\frac{57}{4x-5}.$$

Multiply by $2x+3$, and you will have $45=\frac{114x+171}{4x-5}$; multiply by $4x-5$, and you will have $180x-225=114x+171$; therefore $180x-114x-225=171$, that is, $66x-225=171$; therefore $66x=171+225$, that is, $66x=396$; and $x=6$.

The

The Proof.

The original equation, $\frac{45}{2x+3}=\frac{57}{4x-5}$; $x=6$; therefore $2x=12$; therefore $2x+3=15$; therefore $\frac{45}{2x+3}=\frac{45}{15}=3$: again, $4x=24$; therefore $4x-5=19$; therefore $\frac{57}{4x-5}=\frac{57}{19}=3$; therefore $\frac{45}{2x+3}$ $=\frac{57}{4x-5}$.

Example 12.

$\frac{128}{3x-4}=\frac{216}{5x-6}$: therefore $128=\frac{648x-864}{5x-6}$; therefore $640x$ $-768=648x-864$; therefore $-768=648x-640x-864$, that is, $-768=8x-864$; therefore $+864-768=8x$, that is, $8x=96$; and $x=12$.

The Proof.

The original equation, $\frac{128}{3x-4}=\frac{216}{5x-6}$; $x=12$; therefore $3x$ $=36$; therefore $3x-4=32$; therefore $\frac{128}{3x-4}=\frac{128}{32}=4$: again, $5x=60$; therefore $5x-6=54$; therefore $\frac{216}{5x-6}=\frac{216}{54}=4$; therefore $\frac{128}{3x-4}=\frac{216}{5x-6}$.

Example 13.

$\frac{42x}{x-2}=\frac{35x}{x-3}$: divide both numerators by x, and you will have $\frac{42}{x-2}=\frac{35}{x-3}$; therefore $42=\frac{35x-70}{x-3}$; therefore $42x-126=$ $35x-70$; therefore $42x-35x-126=-70$, that is, $7x-126$ $=-70$; therefore $7x=126-70$, that is, $7x=56$; and $x=8$.

The Proof.

The original equation, $\frac{42x}{x-2}=\frac{35x}{x-3}$; $x=8$; therefore $x-2=6$;

$42x=336$; therefore $\frac{42x}{x-2}=\frac{336}{6}=56$: again, $x-3=5$; and $35x=280$; therefore $\frac{35x}{x-3}=\frac{280}{5}=56$; therefore $\frac{42x}{x-2}=\frac{35x}{x-3}$.

Example 14.

$\frac{xx-12}{3}=\frac{xx-4}{4}$: therefore $xx-12=\frac{3xx-12}{4}$; therefore $4xx-48=3xx-12$; therefore $4xx-3xx-48=-12$, that is, $xx-48=-12$; therefore $xx=+48-12$, that is, $xx=36$; and $x=6$.

The Proof.

The original equation, $\frac{xx-12}{3}=\frac{xx-4}{4}$; $x=6$; therefore $xx=36$; therefore $xx-12=24$; therefore $\frac{xx-12}{3}=\frac{24}{3}=8$: again, $xx-4=32$; therefore $\frac{xx-4}{4}=\frac{32}{4}=8$; therefore $\frac{xx-12}{3}=\frac{x^2-4}{4}$.

Example 15.

$\frac{5xx}{16}-8=12$: therefore $5xx-128=192$; therefore $5xx=192+128$, that is, $5xx=320$; therefore $xx=64$; and $x=8$.

The Proof.

The original equation, $\frac{5xx}{16}-8=12$; $x=8$; therefore $xx=64$; therefore $5xx=320$; therefore $\frac{5xx}{16}=\frac{320}{16}=20$; therefore $\frac{5xx}{16}-8=20-8=12$.

Example 16.

$615x-7xxx=48x$: divide the whole by x, and you will have $615-7xx=48$; therefore $615=7xx+48$; therefore $615-48=7xx$, that is, $7xx=567$; therefore, $xx=81$; and $x=9$.

The Proof.

The original equation, $615x-7xxx=48x$; $x=9$; therefore $xx=81$; therefore $xxx=729$; $7xxx=5103$; again, $615x=5535$; therefore $615x-7xxx=5535-5103=432$: laſtly, $48x=432$; therefore $615x-7xxx=48x$.

THE

THE

ELEMENTS of ALGEBRA

BOOK II.

Preparations for the solution of Algebraic problems.

Article 25. IN solving the following problems, I shall make use of a sort of mixt Algebra, using letters only in representing unknown quantities, and numbers for such as are known. This method, as I take it, will be the best to begin with: but afterwards, when my young scholar has been sufficiently exercised in this way, I shall then introduce him into pure Algebra, which he will find much more extensive than the former, not only as it enables him analytically to find out general solutions, taking in all the particular cases that can be proposed in the problem to which the solution belongs, but also as it enables him afterwards to demonstrate the same solutions or theorems synthetically.

And because I am not yet to suppose him skilled in any of the mathematical sciences, I shall draw my problems, generally speaking, from numbers, either considered abstractedly, or else as they relate to common life.

If a problem be justly proposed, it ought to have as many independent conditions comprehended in it, expressly or implicitly, as there are unknown quantities to be discovered by them; and it must be the chief business of an Algebrist, to search out, sift and distinguish these conditions one from another, before ever he enters upon the solution of his problem.

I said, that so many conditions ought to be comprehended in the problem expressly or implicitly, because it may happen, that a condition may not be expressed in a problem, and yet be implied in the nature of the thing: thus in the 44th problem, where several rods are to be set upright in a streight line, at certain intervals one from another, it is implied, though not expressed, that the number of intervals must be less than the number of rods by unity.

Sometimes a condition may be introduced into a problem, that includes two or more conditions: as when we say, four numbers are in continual proportion, we mean, not only that the first number is to the second as the second is to the third, but also, that the second number is to the third as the third is to the fourth.

Whenever a problem is proposed to be solved algebraically, the Algebrist must substitute some letter of the alphabet for the unknown quantity; and if there be more unknown quantities than one, the rest must receive their names from so many conditions of the problem; and if the problem be justly stated and examined, there will still remain a condition at last, which translated into algebraic language, will afford him an equation, the resolution whereof will give the unknown quantity for which the substitution was made; and when this unknown quantity is once discovered, the rest will be easily discovered by their names. Suppose there are four unknown quantities in a problem; then there ought to be four conditions: now the first unknown quantity receives it's name arbitrarily without any condition; therefore the other three must take up three of the conditions of the problem for their names; and the fourth condition will still be left to furnish out an equation.

The learner must here be very careful to make no positions but what are sufficiently justifiable, either from the express conditions of the problem, or from the nature of the thing; all the liberty he is allowed in cases of this nature is, that he is not obliged to draw out the conditions in the same order as they are given him in the problem, but may make use of them in such an order, as he thinks will be most convenient for his purpose; provided that he does not make use of the same condition twice, except in company with others that have not yet been considered.

My method in the forty four following problems will be, to put down the answer immediately after the problem, and then the solution: for in my opinion, this way of putting down the answer first, will not only serve to illustrate the following solution, but may also serve to fix the problem more firmly in the minds of young beginners, who are but too apt to neglect it, and to substitute chimerical notions of their own, that are not to be justified, either from the conditions of the problem, or common sense.

After the learner has run over some of these problems, and has got a tolerable insight into the method of their resolution, it will be very proper for him to begin again, and to attempt the solution of every problem himself, and not to have recourse to the solutions here given, but in cases of absolute necessity: but after the work is over, he may then compare his own solution with that which is here given, and may alter or reform it as he thinks fit.

The

The solution of some problems producing simple equations.

PROBLEM 1.

26. *What two numbers are those, whose difference is* 14, *and whose sum when added together, is* 48?

Ans. The numbers are 31 and 17: for $31-17=14$; and $31+17=48$.

SOLUTION.

In this problem there are two unknown quantities, to wit, the two numbers sought; and there are two conditions; first, that the less number when subtracted from the greater must leave 14; and secondly, that the two numbers when added together must make 48: therefore I put x for the less number; and to find a name for the greater, I have recourse to the first condition of the problem, which informs me, that the difference betwixt the two numbers sought is 14; therefore, if I call the less number x, I ought to call the greater $x+14$: thus then I have got names for both my unknown quantities, and have still a condition in reserve for an equation, which is the second: now according to this second condition, the two numbers sought, when added together, must make 48; therefore x and $x+14$ when added together must make 48; but x and $x+14$ when added together make $2x+14$; whence I have this equation, $2x+14=48$; therefore $2x=48-14=34$; therefore x, or the less number $=17$, and $x+14$, or the greater number $=31$, as above.

In our solution of this problem, the notation was drawn from the first condition, and the equation from the second; but the notation might have been drawn from the second condition, and the equation from the first, thus: put x for the less number sought; then because the sum of both the numbers is 48, if you subtract the less number x from 48, the remainder $48-x$ will be the greater number, so that the two numbers sought will be x, and $48-x$; subtract the former number from the latter, and the remainder or difference will be $48-2x$; but according to the first condition of the problem, this difference ought to be 14; therefore $48-2x=14$: resolve this equation, and you will have $x=17$, and $48-x=31$, as above.

PROBLEM 2.

27. *Three persons,* A, B *and* C, *make a joint contribution, which in the whole amounts to* 76 *pounds: of this,* A *contributes a certain sum unknown;* B *contributes as much as* A, *and* 10 *pounds more; and* C, *as much*

much as both A *and* B *together: I demand their ſeveral contributions.*

Anſ. A contributes 14 pounds, B 24, and C 38: for $14+10=24$, and $14+24=38$, and $14+24+38=76$.

SOLUTION.

In this problem there are three unknown quantities, and there are three conditions for finding them out; firſt, that the whole contribution amounts to 76 pounds; ſecondly, that B contributes as much as A, and 10 pounds more; and laſtly, that C contributes as much as both A and B together.

Theſe things being ſuppoſed, I firſt put x for A's contribution; then ſince, according to the ſecond condition, B contributes as much as A, and 10 pounds more, I put $x+10$ for B's contribution; laſtly, ſince C contributes as much as both A and B together, I add x and $x+10$ together, and ſo put down the ſum $2x+10$ for C's contribution: thus have I got names for all my unknown quantities, and there remains ſtill one condition unconſidered for my equation, which is, that all the contributions added together make 76 pounds; therefore I add x, and $x+10$, and $2x+10$ together, and ſuppoſe the ſum $4x+20=76$; therefore $4x=76-20=56$; therefore x, or A's contribution equals 14; $x+10$, or B's contribution equals 24; and $2x+10$, or C's contribution equals 38, as above.

PROBLEM 3.

28. *Suppoſe all things as before, except that now, the whole contribution amounts to* 276 *pounds; that of this,* A *contributes a certain ſum unknown; that* B *contributes twice as much as* A, *and* 12 *pounds more; and* C *three times as much as* B, *and* 12 *pounds more: I demand their ſeveral contributions.*

Anſ. A contributes 24 pounds, B 60, and C 192: for $24\times2+12=60$; and $60\times3+12=192$; and $24+60+192=276$.

SOLUTION.

Put x for A's contribution; then becauſe B contributes twice as much as A, and 12 pounds more, B's contribution will be $2x+12$; therefore if C had contributed juſt three times as much as B, his contribution would have amounted to $6x+36$; but according to the problem, C contributes this, and 12 pounds more; therefore C's contribution is $6x+48$; add theſe contributions together, to wit, x, $2x+12$, and $6x+48$, and you will have $9x+60=276$; therefore $9x=276-60=216$; and x, or A's contribution equals 24; whence $2x+12$, or B's

B's contribution equals 60; and $6x+48$, or *C*'s contribution equals 192, as above.

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I know not whether it may not be thought impertinent here to put the learner in mind, that after x was found equal to 24, the other two unknown quantities, $2x+12$, and $6x+48$ were found, by substituting 24 instead of x.

PROBLEM 4.

29. *One begins the world with a certain sum of money, which he improved so well by way of traffick, that at the years end, he found he had doubled his first stock, except an hundred pounds laid out in common expences; and so he went on every year doubling the last years stock, except a hundred a year expended as before; and at the end of three years, found himself just three times as rich as at first: What was his first stock?*

Ans. 140 pounds: for the double of this is 280, and $280-100=180$ pounds at the end of the first year; the double of this last is 360, and $360-100=260$ pounds at the end of the second year; again, the double of this is 520, and $520-100=420$ pounds at the end of the third year; and 420 pounds is just three times as much as 140 pounds, his first stock.

SOLUTION.

Put x for his first stock, that is, let x be the number of pounds he began with; then the double of this is $2x$, and therefore he will have $2x-100$ at the end of the first year; the double of this is $4x-200$; therefore he will have $4x-200-100$, that is, $4x-300$ at the end of the second year; the double of this is $8x-600$; therefore he will have $8x-600-100$, that is, $8x-700$ at the end of the third year; but according to the problem, he ought to have three times his first stock, that is, $3x$, at the end of the third year; therefore $8x-700=3x$; therefore $8x-3x-700=0$, that is, $5x-700=0$; therefore $5x=700$; and x, or his first stock equals 140, as above.

To this problem I shall add another of a like kind, for the learner to solve himself.

One goes with a certain quantity of money about him to a tavern, where he borrows as much as he had then about him, and out of the whole, spends a shilling; with the remainder he goes to a second tavern, where he borrows as much as he had then left, and there also spends a shilling; and so he goes on to a third, and a fourth tavern, borrowing and spend-

ing

ing as before; after which he had nothing left: I demand how much money he had at first about him.

Anf. $\frac{15}{16}$ of one fhilling, that is, 11 pence farthing.

PROBLEM 5.

30. *One has fix fons, each whereof is four years older than his next younger brother; and the eldeft is three times as old as the youngeft: What are their feveral ages?*

Anf. 10, 14, 18, 22, 26, 30: for 30, the age of the eldeft, will then be juft three times 10, that is, three times the age of the youngeft.

SOLUTION.

For their feveral ages put x, $x+4$, $x+8$, $x+12$, $x+16$, $x+20$; then according to the problem $x+20$ the age of the eldeft, ought to be equal to $3x$, that is, three times the age of the youngeft; fince then $3x = x+20$, we fhall have $3x-x=20$, that is, $2x=20$, and $x=10$, as above.

PROBLEM 6.

31. *There is a certain fifh whofe head is 9 inches; the tail is as long as the head and half the back; and the back is as long as both the head and tail together: I demand the length of the back, and of the tail.*

Anf. The length of the back was 36 inches, and that of the tail 27: for $27=9+\frac{36}{2}$; and $36=9+27$.

SOLUTION.

For the length of the back put x; then will x be equal to the length of both head and tail together, by the fuppofition; therefore if from x, the length of the head and tail together, you fubtract 9, the length of the head, there will remain $x-9$ for the length of the tail; but according to the problem, the tail is as long as the head and half the back; therefore $x-9=\frac{x}{2}+9$; therefore $2x-18=x+18$; therefore $2x-x-18=18$, that is, $x-18=18$; and x, the length of the back equals $18+18=36$; therefore $x-9$, the length of the tail equals 27, as above.

PROBLEM 7.

32. *One has a leafe for 99 years; and being afked how much of it was already expired, anfwered, that two thirds of the time paft was equal to four fifths of the time to come: I demand the times paft, and to come.*

Anf.

Anſ. The time paſt was 54 years; and the whole term of years was 99; therefore the time to the expiration of the leaſe was 45 years: now $\frac{2}{3}$ of 54 is 36; and $\frac{4}{5}$ of 45 is 36.

SOLUTION.

Put x for the time paſt; then ſince the whole term of years was 99, if x the time paſt, be ſubtracted from 99 the whole time, there will remain $99-x$ for the time to come; but $\frac{2}{3}$ of the time paſt is $\frac{2x}{3}$; and $\frac{4}{5}$ of the time to come is $\frac{4}{5}$ of $\frac{99-x}{1}=\frac{396-4x}{5}$; therefore $\frac{2x}{3}=\frac{396-4x}{5}$; therefore $2x=\frac{1188-12x}{5}$; therefore $10x=1188-12x$; therefore $10x+12x=1188$, that is, $22x=1188$; and x the time paſt $=54$ years; therefore $99-x$ the time to come equals 45 years.

To this problem I ſhall add two others of the ſame nature, without any ſolution.

Firſt, *To divide the number 84 into two ſuch parts, that three times one part may be equal to four times the other.*

Anſ. The parts are 48 and 36: for in the firſt place, $48+36=84$; and in the next place, three times $48=144=$ four times 36.

Second, *To divide the number 60 into two ſuch parts, that a ſeventh part of one may be equal to an eighth part of the other.*

Anſ. The parts are 28 and 32: for in the firſt place, $28+32=60$; and in the next place, $\frac{1}{7}$ of 28 equals $4=\frac{1}{8}$ of 32.

PROBLEM 8.

33. *It is required to divide the number 50 into two ſuch parts, that $\frac{3}{4}$ of one part being added to $\frac{5}{6}$ of the other, may make 40.*

Anſ. The parts are 20 and 30: for in the firſt place, $20+30=50$; and in the next place, $\frac{3}{4}$ of 20, which is 15, added to $\frac{5}{6}$ of 30, which is 25, makes 40.

SOLUTION.

Put x for one part, and conſequently $50-x$ for the other part; then we ſhall have $\frac{3}{4}$ of $x=\frac{3x}{4}$, and $\frac{5}{6}$ of $50-x=\frac{250-5x}{6}$; but according to the problem, theſe two added together ought to make 40; whence we have this equation, $\frac{3x}{4}+\frac{250-5x}{6}=40$: multiply by 4, and you will

will have $3x + \frac{1000 - 20x}{6} = 160$; multiply again by 6, and you will have $18x + 1000 - 20x = 960$, that is, $1000 - 2x = 960$; therefore $1000 = 2x + 960$; and $1000 - 960 = 2x$, that is, $2x = 40$; and x, which is one of the parts sought, will be 20; whence $50 - x$ or the other part will be 30, as above.

Other two problems of the same nature.

First: *It is required to divide the number* 20 *into two such parts, that three times one part being added to five times the other may make* 84.

Ans. The parts are 8 and 12: for $8 + 12 = 20$; and $8 \times 3 + 12 \times 5$, that is, $24 + 60 = 84$.

Second: *It is required to divide the number* 100 *into two such parts, that if a third part of one be subtracted from a fourth part of the other, the remainder may be* 11.

Ans. The parts are 24 and 76: for first, 24 added to 76 makes 100; and secondly, $\frac{1}{3}$ part of 24, which is 8, subtracted from $\frac{1}{4}$ of 76, which is 19, leaves 11.

PROBLEM 9.

34. *Two persons* A *and* B *engage at play*; A *has* 72 *guineas and* B 52 *before they begin*; *and after a certain number of games won and lost between them*, A *rises with three times as many guineas as* B : *I demand how many guineas* A *won of* B.

Ans. 21: for $72 + 21 = 93$; and $52 - 21 = 31$; and $93 = 31 \times 3$.

SOLUTION.

Put x for the number of guineas A won of B, and consequently that B lost; then will A's last sum be $72 + x$, and B's last sum $52 - x$: now according to the problem, A's last sum is three times as much as B's last sum; that is, three times $\overline{52 - x}$, or $156 - 3x$; therefore $72 + x = 156 - 3x$; therefore $72 + x + 3x = 156$, that is, $72 + 4x = 156$; therefore $4x = 156 - 72 = 84$; therefore x, the money A won of B, equals 21 guineas, as above.

PROBLEM 10.

35. *One meeting a company of beggars, gives to each four pence, and has sixteen pence over*; *but if he would have given them six pence apiece, he would have wanted twelve pence for that purpose* : *I demand the number of persons.*

Ans. 14: for $14 \times 4 + 16 = 72 = 14 \times 6 - 12$.

SOLU-

SOLUTION.

Put x for the number of persons; then if he gives them four pence apiece, the number of pence given will be four times as many as the number of persons, that is, $4x$; therefore $4x+16$ will express all the money he had about him; and so also will $6x-12$ by a like way of reasoning; therefore $4x+16=6x-12$; therefore $16=6x-4x-12=2x-12$; therefore $2x=16+12=28$; and x, the number of persons equals 14, as above.

PROBLEM 11.

36. *What two numbers are those, whose difference is 4, and the difference of whose squares is 112?*

Ans. 12 and 16: for $16-12=4$, and $16\times16-12\times12$, that is, $256-144=112$.

SOLUTION.

The less number, x.		$x+4$
The greater,	$x+4$.	$x+4$
		$xx+4x+16$
		$+4x$
The square of the greater,		$xx+8x+16$
The square of the less,		xx

The difference of their squares, * $8x+16$; whence $8x+16=112$; therefore $8x=112-16=96$; therefore x the less number equals 12, and $x+4$ the greater equals 16, as above.

PROBLEM 12.

37. *What two numbers are those, whereof the greater is three times the less, and the sum of whose squares is five times the sum of the numbers?*

Ans. The numbers are 6 and 2, whose sum is 8: now $6=3$ times 2; and $6\times6+2\times2=40=5$ times 8.

SOLUTION.

The less number,	x.
The greater,	$3x$.
Their sum,	$4x$.
The square of the less,	xx.
The square of the greater,	$9xx$.
The sum of their squares,	$10xx$.

But

But according to the problem, the sum of their squares is 5 times the sum of the numbers, that is, 5 times $4x$ or $20x$; therefore $10xx = 20x$; and $10x = 20$; and x the less number $= 2$; whence $3x$ the greater equals 6, as above.

PROBLEM 13.

38. *What two numbers are those, whereof the less is to the greater as 2 to 3, and the product of whose multiplication is 6 times the sum of the numbers?*

Ans. The numbers are 10 and 15, whose sum is 25: for 10 is to 15 as 2 to 3; this will be plain by putting the question thus; if 2 gives 3, what will 10 give? for the answer will be 15: these numbers will also answer the second condition of the problem; for $10 \times 15 = 150 = 25 \times 6$.

SOLUTION.

Put x for the less number; then to find the greater number say, if 2 gives 3, what will x give? and the answer is $\frac{3x}{2}$; therefore if x stands for the less number, the greater number will be $\frac{3x}{2}$; their sum will be $\frac{x}{1} + \frac{3x}{2}$, or $\frac{2x + 3x}{2}$, or $\frac{5x}{2}$; and the product of their multiplication $x \times \frac{3x}{2}$, or $\frac{3xx}{2}$; but according to the problem, the product of their multiplication ought to be six times the sum of the numbers, that is, six times $\frac{5x}{2}$, or $\frac{30x}{2}$; therefore $\frac{3xx}{2} = \frac{30x}{2}$; and $3x^2 = 30x$; and $3x = 30$; and x the less number equals 10; therefore $\frac{3x}{2}$ the greater number equals 15, as above.

PROBLEM 14.

39. *Two persons* A *and* B *were talking of their money; says* A *to* B, *give me five shillings of your money, and I shall have just as much as you will have left: says* B *to* A, *rather give me five shillings of your money, and I shall then have just three times as much as you will have left: How much money had each?*

Ans. *A* had 15 shillings, and *B* 25: for then, if *A* borrows 5 shillings of *B*, they will have 20 shillings each; on the other hand, if

A lends *B* 5 ſhillings, then will *A* have 10 ſhillings left, and *B* will have 30, which is three times as much.

SOLUTION.

Put x for *A*'s money; then if *A* borrows five ſhillings of *B*, *A* will have $x+5$, and *B* by the ſuppoſition, will have the ſame left, to wit, $x+5$; but if *B* after having lent *A* 5 ſhillings, has $x+5$ left, he muſt have had $x+10$ before; therefore if x repreſents *A*'s money, $x+10$ will repreſent *B*'s: let us now ſuppoſe *B* to borrow 5 ſhillings of *A*; then will *B* have $x+15$, and *A* will have $x-5$; but according to the problem, *B* in this caſe ought to have three times as much as *A* has left, that is, three times $\overline{x-5}$, or $3x-15$; therefore $3x-15=x+15$; therefore $3x-x-15=15$, that is, $2x-15=15$; therefore $2x=15+15=30$; therefore x, or *A*'s money equals 15 ſhillings, and $x+10$, or *B*'s $=25$, as above.

PROBLEM 15.

40. *What two numbers are thoſe, the product of whoſe multiplication is 108, and whoſe ſum is equal to twice their difference?*

Anſ. 18 and 6: for the product of their multiplication is 108; and their ſum 24, is equal to twice their difference 12.

SOLUTION.

For the greater number I put x; then, had their ſum been 108, I ſhould for the other number have put $108-x$; but it is not the ſum of their addition, but the product of their multiplication that is equal to 108; therefore if one number be called x, the other will be $\frac{108}{x}$, which I thus demonſtrate: let y be the other number; then will $x\times y$ or $xy=108$ by the ſuppoſition; divide both ſides of the equation by x, and you will have $y=\frac{108}{x}$; as was to be demonſtrated. This being admitted, the difference between the greater number x, and the leſs $\frac{108}{x}$, is $x-\frac{108}{x}$; and their ſum is $x+\frac{108}{x}$: but by the condition of the problem, this ſum ought to be equal to twice the difference, that is, to twice $\overline{x-\frac{108}{x}}$ or $2x-\frac{216}{x}$; therefore $2x-\frac{216}{x}=x+\frac{108}{x}$; therefore $2xx$

$-216 = xx + 108$; therefore $2xx - xx - 216 = 108$, that is, $xx - 216 = 108$; therefore $xx = 108 + 216 = 324$; therefore x the greater number equals 18, and $\frac{108}{x}$ the less equals 6, as above.

PROBLEM 16.

41. *It is required to divide the number 48 into two such parts, that one part may be three times as much above 20, as the other wants of 20.*

Ans. The two parts are 32 and 16: for $32 + 16 = 48$; moreover 32 is 12 above 20, and 16 wants 4 of 20, and 12 is three times 4.

SOLUTION.

Put x for the less number sought; then will $48 - x$ be the greater, and the excess of this greater above 20 will be $28 - x$, as is evident by subtracting 20 from $48 - x$: again, the excess of 20 above the less number (which is, what the less number wants of 20) is $20 - x$; and according to the problem, the former excess is three times the latter, that is, three times $\overline{20 - x}$, or $60 - 3x$; whence we have this equation, $28 - x = 60 - 3x$; therefore $28 - x + 3x = 60$, that is, $28 + 2x = 60$; therefore $2x = 60 - 28 = 32$; therefore x the less part $= 16$, and $48 - x$ the greater $= 32$, as above.

Another solution of the foregoing problem.

Put x for what the less number wants of 20; then will the less number be $20 - x$, the greater $20 + 3x$, and their sum $40 + 2x$; but by the problem, their sum is 48; therefore $40 + 2x = 48$; therefore $2x = 48 - 40 = 8$; therefore $x = 4$; whence $20 - x$ the less number $= 16$, and $20 + 3x$ the greater $= 32$.

PROBLEM 17.

42. *One has three debtors,* A, B *and* C, *whose particular debts he has forgot; but thus much he could remember from his accounts, that* A's *and* B's *debts together amounted to* 60 *pounds;* A's *and* C's *to* 80 *pounds; and* B's *and* C's *to* 92 *pounds: I demand the particulars.*

Ans. *A*'s debt was 24 pounds, *B*'s 36, and *C*'s 56: for $24 + 36 = 60$, $24 + 56 = 80$, and $36 + 56 = 92$.

SOLUTION.

Put x for *A*'s debt; then because *A*'s and *B*'s together made 60 pounds, *B*'s debt will be $60 - x$: again, because *A*'s and *C*'s together made 80 pounds,

pounds, *C*'s debt must be $80 - x$: now since according to the problem, *B*'s and *C*'s debts when added together make 92 pounds, I add $60 - x$, and $80 - x$ together, and suppose the sum $140 - 2x = 92$; whence $2x + 92 = 140$; and $2x = 140 - 92 = 48$; and x, that is, *A*'s debt $= 24$ pounds; whence $60 - x$, or *B*'s debt $= 36$ pounds; and $80 - x$, or *C*'s, is 56 pounds, as above.

PROBLEM 18.

43. *One being asked how many teeth he had remaining in his head, answered, three times as many as he had lost; and being asked how many he had lost, answered, as many as being multiplied into $\frac{1}{6}$ part of the number left, would give all he ever had at first: I demand how many he had lost, and how many he had left.*

Ans. He had lost 8, and had 24 left: for then 24 the number left, will be equal to 3 times 8, the number lost; and moreover 8 the number lost, multiplied into 4, that is, into $\frac{1}{6}$ part of 24 the number left, will give $32 = 24 + 8$, all he ever had at first.

SOLUTION.

Teeth lost, x.
left, $3x$.
In all, $4x$.

$\frac{1}{6}$ part of the number left $\frac{3x}{6}$, or $\frac{x}{2}$; this multiplied into the number lost, makes $\frac{x}{2} \times x$ or $\frac{xx}{2}$; but according to the problem, this product is equal to all he ever had at first; whence $\frac{xx}{2} = 4x$; and $xx = 8x$; and x, the number lost $= 8$; whence $3x$, the number left $= 24$, as above.

PROBLEM 19.

44. *One rents 25 acres of land at 7 pounds 12 shillings* per annum; *which land consists of two sorts, the better sort he rents at 8 shillings per acre, and the worse at 5: I demand the number of acres of each sort.*

Ans. He had 9 acres of the better sort, and 16 of the worse: for $9 + 16$ acres $= 25$ acres; and 9 times 8 shillings $= 72$ shillings; and 16 times 5 shillings $= 80$ shillings; and $72 + 80 = 152$ shillings $= 7$ pounds 12 shillings.

SOLU-

SOLUTION.

Put x for the number of acres of the better ſort; then will $25-x$ be the number of acres of the worſe ſort, becauſe both together made 25 acres: moreover, ſince he paid 8 ſhillings an acre for the better ſort, he muſt pay 8 times as many ſhillings as he had acres, that is, $8x$; and ſince he paid 5 ſhillings an acre for the worſe ſort, he muſt pay 5 times as many ſhillings as he had acres of this ſort, that is, $\overline{25-x}\times 5$, or $125-5x$: put both theſe rents together, and they will amount to $8x+125-5x$, or $3x+125$ in ſhillings; but they amount to 152 ſhillings, by the ſuppoſition; therefore $3x+125=152$; therefore $3x=152-125=27$; therefore x, the number of acres of the better ſort $=9$, and $25-x$, the number of the worſe ſort $=16$, as above.

PROBLEM 20.

45. *One hires a labourer into his garden for 36 days upon the following conditions, to wit, that for every day he laboured, he was to receive two ſhillings and ſixpence; and for every day he was abſent, he was to forfeit one ſhilling and ſixpence: now at the end of the 36 days, after due deductions made for his forfeitures, he received clear 2 pounds 18 ſhillings: I demand how many days he laboured, and how many he was abſent.*

Anſ. He laboured 28 days, and loitered 8: for 28 half-crowns amount to 3 pounds 10 ſhillings due to him for wages; and 8 eighteenpences amount to 12 ſhillings due from him in forfeitures; and this latter ſum ſubtracted from the former, leaves 2 pounds 18 ſhillings to be received clear.

SOLUTION.

Put x for the number of days he laboured; then will $36-x$ repreſent the number of days he was abſent: again, ſince he was to receive 30 pence for every day he laboured, the pence due to him in wages will be $30\times x$, or $30x$; and ſince he was to forfeit 18 pence for every day he was abſent; the pence due from him in forfeitures will be $18\times\overline{36-x}$, or $648-18x$: ſubtract now $648-18x$, the pence due from him in forfeitures, from $30x$, the pence due to him for wages; or, which is all one, add $18x-648$ to $30x$, and there ariſes $48x-648$, the pence to be received clear: but he received clear 2 pounds 18 ſhillings, or 696 pence, by the ſuppoſition; therefore $48x-648=696$; therefore $48x=648+696=1344$; therefore x, the number of days he laboured $=28$; and $36-x$, the number of days he loitered $=8$, as above.

PRO-

PROBLEM 21.

46. *Suppoſe that* 19 *pounds of gold weigh* 18 *pounds in water; and moreover, that* 10 *pounds of ſilver weigh* 9 *pounds in water; laſtly, ſuppoſe a maſs weighing* 106 *pounds, and conſiſting of both gold and ſilver, when weighed in water, weighs only* 99 *pounds: I demand the diſtinct quantities of gold and ſilver in the maſs.*

Anſ. There were 76 pounds of gold, and 30 of ſilver: for $76+30=106$; moreover, if 19 pounds of gold weigh 18 pounds in water, 76 pounds of gold will weigh 72 pounds in water, by the golden rule; and if 10 pounds of ſilver weigh 9 pounds in water, 30 pounds of ſilver will weigh 27 pounds in water, by the ſame rule; and laſtly $72+27$, the weight of the whole maſs in water $=99$, as the problem requires.

SOLUTION.

Put x for the number of pounds of gold in the maſs; then will $106-x$ repreſent the number of pounds of ſilver; and to find the weight of the former in water, I ſay, if 19 pounds of gold weigh 18 pounds in water, what will x weigh? and the anſwer is $\frac{18x}{19}$: again, to find the weight of the ſilver in water, I ſay, if 10 pounds of ſilver weigh 9 pounds in water, what will $106-x$ weigh? and the anſwer is $\frac{954-9x}{10}$; then I add both theſe weights together, and their ſum is $\frac{18x}{19}+\frac{954-9x}{10}$; but according to the problem, the weight of the whole maſs in water is 99 pounds; therefore the equation is $\frac{18x}{19}+\frac{954-9x}{10}=99$; therefore $18x+\frac{18126-171x}{10}=1881$; therefore $180x+18126-171x=18810$, that is, $9x+18126=18810$; therefore $9x=18810-18126=684$; therefore x, the pounds of gold in the maſs $=76$; and $106-x$, the pounds of ſilver $=30$, as above.

Note. In the ſolution of this problem it is taken for granted, that all bodies of the ſame ſpecific gravity, have their weights in water proportionable to their weights in air, which is eaſily demonſtrated from hydroſtatics, and is further confirmed by experiments: thus if the weight of any one quantity of gold in air be to it's weight in water as 19 to 18,

the

the weight of any other quantity of gold in air will be to it's weight in water in the ſame proportion; and the ſame may be obſerved of ſilver; only as ſilver is a body of a different ſpecific gravity from gold, the proportion will be different, being as 10 to 9, and ſometimes as 11 to 10.

PROBLEM 22.

47. *One lets out a certain ſum of money at* 6 per cent, *ſimple intereſt, which intereſt in* 10 *years time wanted but* 12 *pounds of the principal: What was the principal?*

Anſ. The principal was 30 pounds, and the intereſt 18 pounds $=30-12$: for as 100 pounds principal is to it's annual intereſt 6 pounds, ſo is 30 pounds principal to its annual intereſt 1.8 pounds; and therefore it's 10 years intereſt will be 18 pounds.

SOLUTION.

Put x for the number of pounds in the principal; then to find it's intereſt for one year, ſay, if 100 pounds principal give 6 pounds intereſt, what will x principal give? and the anſwer will be $\frac{6x}{100}$; this will be the intereſt of x for one year, and therefore it's intereſt for 10 years will be $\frac{60x}{100}$, or $\frac{6x}{10}$, or $\frac{3x}{5}$: but according to the problem, this intereſt is to be $x-12$; for it is to want juſt 12 pounds of the principal, by the ſuppoſition; therefore $x-12=\frac{3x}{5}$; therefore $5x-60=3x$; therefore $5x-3x-60=0$, that is, $2x-60=0$; therefore $2x=60$, and x the principal $=30$, and $\frac{3x}{5}$ the 10 years intereſt $=18$ pounds, as above.

PROBLEM 23.

48. *One lets out* 98 *pounds in two different parcels; one at* 5, *the other at* 6 per cent, *ſimple intereſt; and the intereſt of the whole in* 16 *years amounted to* 81 *pounds: What were the two parcels?*

Anſ. The parcel at 5 *per cent* was 48 pounds, and the other at 6 *per cent* was 50 pounds: for in the firſt place, $48+50=98$; and moreover, the annual intereſt of 48 pounds at 5 *per cent* amounts to 2 pounds 8 ſhillings; and the annual intereſt of 50 pounds at 6 *per cent* is 3 pounds; therefore the whole intereſt amounts to 5 pounds 8 ſhillings in one year; and conſequently to 81 pounds in 15 years.

SOLU-

SOLUTION.

Put x for the number of pounds in the parcel at 5 *per cent*, and consequently $98 - x$ for the number of pounds in the other parcel at 6 *per cent*; then to find the annual intereſt of x, ſay, if 100 pounds principal give 5 pounds intereſt, what will x give? and the anſwer will be $\frac{5x}{100}$: again, for the other parcel, ſay, if 100 pounds principal give 6 pounds intereſt, what will $98 - x$ give? and the anſwer will be $\frac{588 - 6x}{100}$: add theſe two intereſts together, to wit, $\frac{5x}{100}$ and $\frac{588 - 6x}{100}$, and the ſum will be $\frac{5x + 588 - 6x}{100}$, that is, $\frac{588 - x}{100}$; this is the intereſt of the two parcels for one year; and therefore in 15 years time, this intereſt muſt amount to $\frac{8820 - 15x}{100}$; but it amounts to 81 pounds, by the ſuppoſition; therefore $\frac{8820 - 15x}{100} = 81$; therefore $8820 - 15x = 8100$; therefore $8820 = 15x + 8100$; therefore $15x = 8820 - 8100 = 720$; therefore x, the parcel at 5 *per cent* $= 48$ pounds; and $98 - x$, the parcel at 6 *per cent* $= 50$ pounds, as above.

PROBLEM 24.

49. *A gentleman hires a ſervant for a year, or 12 months, and was to allow him for his wages ſix pounds in money, together with a livery cloak of a certain value agreed upon: but after ſeven months, upon ſome miſdemeanour of the ſervant, he turns him off, with the aforeſaid cloak and 50 ſhillings in money; which was all that was due to him for that time: I demand the value of the cloak.*

Anſ. The value of the cloak was 48 ſhillings: for then his whole wages for 12 months would be 168 ſhillings; and by the rule of proportion, his wages for 7 months would be 98 ſhillings; whence ſubtracting 48 ſhillings, the value of the cloak, there would remain 50 ſhillings due to him in money.

SOLUTION.

Put x for the value of the cloak in ſhillings; then will his whole wages for 12 months be $x + 120$; and his wages for 7 months, may be found by the golden rule, ſaying, as 12 is to 7, ſo is $x + 120$ to

$\frac{7x+840}{12}$; but according to the problem, his wages for 7 months was the cloak and 50 ſhillings in money, that is, $x+50$; therefore $x+50=\frac{7x+840}{12}$; therefore $12x+600=7x+840$; therefore $12x-7x+600=840$, that is, $5x+600=840$; therefore $5x=840-600=240$; therefore x, the value of the cloak in ſhillings, is 48, as above.

PROBLEM 25.

50. *One diſtributes 20 ſhillings among 20 people, giving 6 pence apiece to ſome, and 16 pence apiece to the reſt: I demand the number of perſons of each denomination.*

Anſ. There were 8 perſons who received 6 pence apiece; and 12 who received 16 pence apiece: for in the firſt place, $8+12=20$ perſons; and ſince 8 ſixpences are equivalent to 4 ſhillings, and 12 ſixteenpences to 16 ſhillings, we ſhall have in the next place, $4+16=20$ ſhillings.

SOLUTION.

Put x for the number perſons who received 6 pence apeice; then ſince there were 20 perſons in all, $20-x$ will be the number of thoſe who received ſixteenpence apiece: the number of pence received by the former company will be $6x$; and the number of pence received by the latter will be $\overline{20-x}\times16$, that is, $320-16x$; and therefore the whole number of pence received will be $6x+320-16x$, or $320-10x$; but according to the problem, there was received in the whole, 20 ſhillings, or 240 pence; therefore, $320-10x=240$; therefore $10x+240=320$; therefore $10x=320-240=80$; therefore x, the number of perſons who received ſixpence a piece, is 8, and conſequently $20-x$, the number of the reſt is 12, as above.

PROBLEM 26.

51. *It is required to divide 24 ſhillings into 24 pieces, conſiſting only of ninepences and thirteenpencehalfpennies.*

Anſ. There muſt be 8 ninepences and 16 thirteenpencehalfpennies: for in the firſt place, $8+16=24$ pieces; and ſince 8 ninepences are equivalent to 6 ſhillings, and 16 thirteenpencehalfpennies to 18 ſhillings, we have in the next place $6+18=24$ ſhillings.

SOLU-

SOLUTION.

Put x for the number of ninepences, and consequently $24-x$ for the number of thirteenpencehalfpennies: now the number of halfpence equivalent to the former is $18x$, because there are 18 halfpence in every 9 pence; and the number of halfpence equivalent to the latter is $\overline{24-x}$ $\times 27$, or $648-27x$, because there are 27 halfpence in every thirteenpencehalfpenny piece; therefore the number of halfpence equivalent to the whole will be $18x+648-27x$, that is, $648-9x$; but according to the problem, the whole amounts to 24 shillings, or 576 halfpence; therefore $648-9x=576$; therefore $9x+576=648$; therefore $9x=648-576=72$; therefore x, the number of ninepences is 8; and $24-x$, the number of thirteenpencehalfpennies is 16, as above.

PROBLEM 27.

52. *Two persons,* A *and* B, *travelling together,* A *with* 100, *and* B *with* 48 *pounds about him, met a company of robbers, who took twice as much from* A *as from* B, *and left* A *thrice as much as they left* B: *I demand how much they took from each.*

Ans. They took 44 pounds from *B*, and twice as much, that is, 88 pounds from *A*, so they left *B* 4 pounds, and *A* 12 pounds, which is 3 times 4.

SOLUTION.

Taken from B, x.
from A, $2x$.
Left B, $48-x$.
Left A, $100-2x$.

But according to the problem, they left A three times as much as they left B, that is, three times $48-x$, or $144-3x$; therefore $100-2x=144-3x$; therefore $100-2x+3x=144$, that is, $100+x=144$; therefore x, the sum taken from $B=144-100=44$; and $2x$, or 88 is the sum taken from A, as above.

PROBLEM 28.

53. *A certain cistern which would be filled in* 12 *minutes of time by two pipes running into it, would be filled in* 20 *minutes by one alone: I demand in what time it would be filled by the other alone.*

Ans. In 30 minutes: for at this rate, this pipe would discharge $\frac{1}{30}$ part of a cisternful in one minute's time, and $\frac{12}{30}$ or $\frac{2}{5}$ in 12 minutes: but the other pipe in 12 minutes time discharges $\frac{12}{20}$ or $\frac{3}{5}$ of a cisternful, by

 the

the ſuppoſition; therefore both pipes in 12 minutes will fill $\frac{5}{5}$, that is, the whole ciſtern.

SOLUTION.

Put x for the time ſought wherein the ſecond pipe will fill the ciſtern; then to find how much of this ciſtern will be filled by that pipe in 12 minutes time, ſay by the golden rule, if in the time x be diſcharged one whole ciſternful, how much of it will be diſcharged in 12 minutes? and the anſwer will be $\frac{12}{x}$; and for the ſame reaſon $\frac{12}{20}$ will be the part filled by the firſt pipe in the ſame time of 12 minutes; and $\frac{12}{x}+\frac{12}{20}$ will be the quantity they will both diſcharge in this time; but according to the problem, they ought both together to diſcharge one whole ciſternful in this time: therefore $\frac{12}{x}+\frac{12}{20}=1$; therefore $12+\frac{12x}{20}=x$; therefore $240+12x=20x$; therefore $20x-12x$, or $8x=240$; and x, the time wherein the ſecond pipe alone would fill the ciſtern, is 30 minutes, as above.

PROBLEM 29.

54. *A general of an army diſpoſing his men in rank and file in form of a ſquare, found he had 60 men more than would ſtand in the figure; but when he thought of enlarging the ſide of his ſquare, though but by one man more, he found he ſhould want 41 men for that purpoſe: I demand his number of men, and how many they ſtood of a ſide.*

Anſ. He had 2560 men, and they ſtood 50 of a ſide: for the better underſtanding whereof it muſt be obſerved, that ſoldiers are ſaid to ſtand in rank and file, when a rank is placed behind the foremoſt, and then another behind that, and ſo on: thus if there be 6 ranks, and 7 men in every rank, they are ſaid to be 7 in rank and 6 in file; and the number of men ſo diſpoſed are 6 times 7, or 42. This being underſtood, let us now ſuppoſe the general to have 50 ranks with 50 men in every rank; then they would conſtitute a ſquare whoſe ſide is 50, and the number of men ſo diſpoſed would be 50 times 50, or 2500; therefore if we ſuppoſe him to have 2560 men in all, he would have 60 men more than could ſtand in this figure: ſuppoſe then that the ſide of his ſquare was enlarged from 50 to 51 men, that is, let us ſuppoſe 51 ranks with 51 men in every rank; then it is plain, that this figure would take up 51 times 51 or 2601; but he had only 2560 men; therefore this figure would

would take up more than he had by 41; for 2560 ſubtracted from 2601 leaves 41.

SOLUTION.

Put x for the number of men that conſtituted the ſide of the ſquare; then will $x\times x$ or xx be the number of men the figure took up; but according to the problem, he had 60 men more; therefore $xx+60$ will repreſent his whole number of men: ſuppoſe now he had $x+1$ men of a ſide, then $\overline{x+1}\times\overline{x+1}$, or $xx+2x+1$ will be the number of men this ſuppoſed figure would have taken up; but according to the problem, this laſt number $xx+2x+1$ muſt be greater than the true number of men $xx+60$ by 41; therefore $xx+60$ ſubtracted from $xx+2x+1$, ought to leave 41; but $xx+60$ ſubtracted from $xx+2x+1$ leaves $+2x-59$; therefore $2x-59=41$; therefore $2x=41+59=100$; therefore x, the number of men of a ſide $=50$; and $xx+60$, the whole number of men $=2560$, as above.

$$\begin{array}{l} x+1 \\ x+1 \\ \hline xx+x+1 \\ \quad +x \\ \hline xx+2x+1. \end{array}$$

$$\begin{array}{l} xx+2x+1 \\ xx \quad * +60 \\ \hline * +2x-59. \end{array}$$

PROBLEM 30.

55. *There are two places 154 miles diſtant from each other; from whence two perſons ſet out at the ſame time with a deſign to meet, one travelling at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours: I demand how long and how far each travelled before they met.*

Anſ. As our travellers were ſuppoſed both to ſet out at the ſame time, and they muſt both meet at the ſame time, it follows, that each muſt perform his journey in the ſame time; I ſay then, that each performed his journey in 56 hours: for if in 2 hours the firſt travelled 3 miles, in 56 hours he muſt travel 84 miles, by the rule of proportion; in like manner, if in 4 hours the ſecond travels 5 miles, in 56 hours he muſt travel 70 miles; and $84+70=154$ miles, the whole diſtance.

SOLUTION.

Put x for the number of hours each travelled; then to find how many miles the firſt travelled, ſay, if in 2 hours he travelled 3 miles, how far did he travel in x hours? and the anſwer is $\frac{3x}{2}$; then for the other ſay,

if

if in 4 hours he travelled 5 miles, how far did he travel in x hours? and the anſwer is $\frac{5x}{4}$; therefore both their journies put together make $\frac{3x}{2}+\frac{5x}{4}$; but they both travelled the whole diſtance, 154 miles; therefore $\frac{3x}{2}+\frac{5x}{4}=154$; therefore $3x+\frac{10x}{4}=308$; therefore $12x+10x$, that is, $22x=1232$; therefore x, the number of hours each travelled $=56$; therefore $\frac{3x}{2}$, the number of miles the firſt travelled $=84$; and $\frac{5x}{4}$, the number of miles the ſecond travelled $=70$, as above.

PROBLEM 31.

56. *One ſets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours after, another ſets out from the ſame place, and travels the ſame road at the rate of 5 miles in 3 hours: I demand how long and how far the firſt muſt travel before he is overtaken by the ſecond.*

Anſ. The firſt muſt travel 50 hours and conſequently 70 miles; the ſecond muſt travel 50—8, or 42 hours, and conſequently alſo 70 miles: ſince then they both ſet out from the ſame place, and the ſecond traveller has now travelled as far as the firſt, he muſt have overtaken the firſt.

SOLUTION.

Put x for the number of hours the firſt travelled, and conſequently $x-8$ for the number of hours wherein the ſecond travelled: then to find the miles travelled by the firſt, ſay, if in 5 hours he travels 7 miles, how far will he travel in x hours? and the anſwer is $\frac{7x}{5}$; then for the other ſay, if in 3 hours he travelled 5 miles, how far will he travel in $x-8$ hours, and the anſwer is $\frac{5x-40}{3}$; but as theſe two travellers both ſet out from the ſame place, and muſt come together at the ſame place, it follows, that they muſt both travel the ſame length of ſpace; therefore $\frac{5x-40}{3}=\frac{7x}{5}$; therefore $5x-40=\frac{21x}{5}$; therefore $25x-200=21x$; therefore $25x-21x-200=0$, that is, $4x-200=0$; therefore $4x=200$; and x, the hours travelled by the firſt $=50$;
whence

whence $x-8$, the hours travelled by the fecond $=42$; $\frac{7x}{5}$, the miles travelled by the firft $=70$; and $\frac{5x-40}{3}$, the miles travelled by the fecond $=70$, as above.

PROBLEM 32.

57. *One looking upon a clock, and being afked what a clock it was, anfwered, between 5 and 6; but a more particular anfwer being defired, he faid, that at that very moment, the minute hand and the hour hand were together: what was the time of the day?*

Anf. Juft $\frac{5}{11}$ of an hour, or 27 minutes and about 16 feconds paft 5.

For a clearer and more diftinct explication of this anfwer, and of the following folution, let us call that point upon the dial plate wherein the two hands are fuppofed to exift together at the inftant fought, between the hours of 5 and 6, *A*; then, fince the hour hand moves from 12 to 5 in 5 hours, if the time of $\frac{5}{11}$ of an hour be rightly affigned, it ought to move from 12 to *A* in 5 hours and $\frac{5}{11}$ of an hour: again, at 5 a clock the minute hand pointed at 12, and according to this account, $\frac{5}{11}$ of an hour after, it pointed at *A*; therefore the minute hand moves from 12 to *A* in $\frac{5}{11}$ of an hour; this then being the cafe, to wit, that the hour hand moves from 12 to *A*, in 5 hours and $\frac{5}{11}$ of an hour, and that the minute hand paffes over the fame fpace in $\frac{5}{11}$ of an hour, let us in the next place enquire how thefe times agree with the known velocities of thefe two hands; for the minute hand makes a revolution in an hour's time, and the hour hand in 12 hours; therefore the hour hand moves 12 times flower than the minute hand; therefore the hour hand ought to be 12 times as long in paffing over any given fpace, as the minute hand is in paffing through the fame fpace; but according to our account, the hour hand paffed from 12 to *A* in 5 hours and $\frac{5}{11}$ of an hour, and the minute hand paffed over the fame fpace in $\frac{5}{11}$ of an hour; therefore if the time enquired after be truly affigned, 5 hours and $\frac{5}{11}$ of an hour ought to be 12 times as much as $\frac{5}{11}$ of an hour; and fo we find it; for 12 times $\frac{5}{11}=\frac{60}{11}=5\frac{5}{11}$.

SOLUTION.

Put x for the part or parts of an hour from 5 a clock to the inftant fought; then will the hour hand move from 12 to *A*, in $x+5$ hours, and the minute hand will pafs over the fame fpace in the time x; and $x+5$ will be to x as 12 to 1; all which are manifeft from what has been faid above; multiply the extremes and middle terms of this laft proportion

proportion together, according to the 15th article, and you will have $12x = x + 5$ hours; and $12x - x$, that is, $11x = 5$ hours; and $x = \frac{5}{11}$ of an hour, as above.

PROBLEM 33.

58. *A clock has two hands turning upon the same center, whereof the swifter makes a revolution every 12 hours, and the slower every 16 hours: I demand the synodical period of these two hands.*

Ans. 48 hours.

By a synodical period is here meant, the time that passes from the instant the two hands are together, to the instant they next come together again: but to acquire a more distinct idea of this time, that is, such a one as we may form a calculation upon, let us suppose the two hands together; then it is plain, that the swifter hand will immediately get before the slower; and in time, will have got a quarter of a circle, and half a circle, and three quarters of a circle, and at last an entire circle before the slower, in which case the two hands will now be together again; and this will be the first time of their coming together again since the supposed time of their setting out; therefore a synodical period may now be defined to be the time wherein the swifter hand gets an entire circle of the slower, or makes one revolution more than the slower: since then in 48 hours the slower hand makes 3 revolutions, and the swifter 4, it follows, that 48 hours must be their true synodical period.

SOLUTION.

Put x for the number of hours in a synodical period; then to find the number of revolutions made by the slower hand in the time x, say, if in 16 hours the slower hand makes one revolution, how many will it make in the time x? and the answer will be $\frac{x}{16}$; then for the swifter hand say, if in 12 hours it makes one revolution, how many will it make in the time x? and the answer will be $\frac{x}{12}$; therefore $\frac{x}{16}$ is the number of revolutions made by the slower hand, and $\frac{x}{12}$ the number of revolutions made by the swifter hand in the same time x: now to know how much the swifter hand has got of the slower, I subtract the former space $\frac{x}{16}$, from the latter $\frac{x}{12}$, and the remainder is $\frac{x}{12} - \frac{x}{16}$, or $\frac{x}{48}$; but as x is the time of a synodical period, it is plain from what has been said, that the swifter hand ought to have got one entire circle or revolution

lution before the ſlower; therefore $\frac{x}{48} = 1$; and x, the number of hours in a ſynodical period $= 48$, as above.

PROBLEM 34.

59. *A vintner has two ſorts of wines, one ſort worth* 20 *pence a quart, and the other worth* 12 *pence a quart; out of which he wants a mixture of a* 100 *quarts worth* 14 *pence the quart: the queſtion is, how many quarts he muſt mix of each ſort.*

Anſ. There muſt muſt be 25 quarts of the better ſort, and 75, that is, $100 - 25$ of the worſe: for 25 quarts at 20 pence a quart amount to 500 pence; and 75 quarts at 12 pence a quart amount to 900 pence; therefore the whole amounts to 1400 pence; which divided by 100, the number of quarts, gives 14 pence a quart.

SOLUTION.

Put x for the number of quarts of the better ſort, and conſequently $100 - x$ for thoſe of the worſe; then will $20x$ be the price in pence of the better ſort, $1200 - 12x$ that of the worſe, and $8x + 1200$ that of the whole: but 100 quarts at 14 pence a quart amount to 1400 pence; therefore $8x + 1200 = 1400$; $8x = 200$; and x, the number of quarts of the better ſort $= 25$; and $100 - x$, the number of quarts of the worſe ſort $= 75$, as above.

PROBLEM 35.

60. *It is required to divide the number* 90 *into* 4 *ſuch parts, that the firſt part encreaſed by* 2, *the ſecond diminiſhed by* 2, *the third multiplied by* 2, *and the fourth divided by* 2, *may be all equal.*

Anſ. The parts are 18, 22, 10, and 40: for $18 + 22 + 10 + 40 = 90$; and $18 + 2 = 22 - 2 = 10 \times 2 = \frac{40}{2} = 20$.

SOLUTION.

1ſt, Put x for the firſt part; then if this firſt part be encreaſed by 2, the ſum will be $x + 2$; therefore the ſecond part when diminiſhed by 2, the third when multiplied by 2, and the fourth when divided by 2, ought each to make $x + 2$.

2dly, But if the ſecond part when diminiſhed by 2 be $x + 2$, then before it was diminiſhed it was $x + 4$.

3dly, If the third part when doubled or multiplied by 2 be $x + 2$, then before it was multiplied, it muſt be the half of $x + 2$, that is, $\frac{x}{2} + 1$.

 Laſtly,

Lastly, If half the fourth number be $x+2$, the fourth number itself must be $2x+4$; so that at last, the parts are found to be x, $x+4$, $\frac{x}{2}+1$, and $2x+4$; add these parts all together, and the sum will be $4x+\frac{x}{2}+9$; whence we have this equation, $4x+\frac{x}{2}+9=90$; therefore $8x+x+18=180$; therefore $9x=162$; and x, the 1st part $=18$; whence $x+4$, the 2d part $=22$; and $\frac{x}{2}+1$, the 3d part $=10$; and $2x+4$, the 4th part $=40$, as above.

PROBLEM 36.

61. *A shepherd driving a flock of sheep in time of war, meets a company of soldiers who plunder him of half his flock, and half a sheep over; the same treatment he meets with from a second, a third and a fourth company, every succeeding company plundering him of half the flock the last had left, and half a sheep over, insomuch that at last he had but 7 sheep left: I demand how many he had at first.*

Ans. His flock at first consisted of 127 sheep; and if the first company had only robbed him of half his flock, they would have left him $63\frac{1}{2}$ sheep; but as they plundered him of half his flock, and half a sheep over, they left him only 63 sheep; in like manner the second company left him 31, the third 15, and the fourth 7.

N. B. Before I enter upon the solution of this problem, I must put the learner in mind of what he has been told before, (introduction art. 13.) to wit, that a fraction may be halved two ways, either by halving the numerator, or doubling the denominator.

SOLUTION.

Put x for the number of his first flock; then had the first company only taken half his flock, they would have left him the other half, *viz.* $\frac{x}{2}$; but they took half his flock and half a sheep over; therefore they left him just so much less, to wit, $\frac{x}{2}-\frac{1}{2}$, or $\frac{x-1}{2}$: again, had the second company only taken half what remained, they would have left him half, to wit, $\frac{x-1}{4}$; but by taking half a sheep more, they left him $\frac{x-1}{4}-\frac{1}{2}$, that is, $\frac{2x-2-4}{8}$, or $\frac{2x-6}{8}$, or $\frac{x-3}{4}$; in like manner

ner the third company left $\frac{x-3}{8}-\frac{1}{2}$, or $\frac{2x-6-8}{16}$, or $\frac{2x-14}{16}$, or $\frac{x-7}{8}$; and the last company left him $\frac{x-7}{16}-\frac{1}{2}$, or $\frac{x-15}{16}$; but they left him 7 sheep, by the supposition; therefore $\frac{x-15}{16}=7$; and $x-15=112$; and x his first number $=127$, as above.

PROBLEM 37.

62. *One buys a certain number of eggs, half whereof he buys in at 2 a penny, and the other half at three a penny; these he afterwards sold out again at the rate of 5 for twopence, and contrary to his expectation, lost a penny by the bargain: what was the number of his eggs?*

Ans. The number of his eggs was 60; half whereof at two a penny cost him 15 pence; and the other half at three a penny, ten pence; and the whole 25 pence: but 60 eggs sold out at 5 for two pence, would only bring him in 24 pence, as appears by the rule of proportion; therefore he lost a penny by the bargain.

SOLUTION.

Put x for the number of eggs; then say, if 2 eggs cost one penny, what will $\frac{x}{2}$ one half of his eggs cost? and the answer will be $\frac{x}{4}$; and for the same reason the other half at three a penny will cost him $\frac{x}{6}$; so that for the whole he must pay $\frac{x}{4}+\frac{x}{6}$, or $\frac{5x}{12}$: again say, if 5 eggs were sold for two pence, what were x eggs sold for? and the answer will be $\frac{2x}{5}$; therefore $\frac{2x}{5}$ will be the number of pence he received for his eggs; subtract this from $\frac{5x}{12}$, the pence he paid for them, and the remainder $\frac{5x}{12}-\frac{2x}{5}$, or $\frac{x}{60}$ will be his loss; but by the supposition, he lost one penny; therefore $\frac{x}{60}=1$; and x the number of eggs will be 60, as above.

PROBLEM 38.

63. *One draws a certain quantity of wine out of a full vessel that held 81 gallons; and then recruiting the vessel with water, takes a second*

draught of as much wine and water together, as before he did of wine; and so he goes on for four draughts one after another, always taking the same quantity at a draught, and then recruiting the vessel with water; insomuch that at last, there were not above 16 gallons of pure wine left in the vessel, all the rest being water: I demand how much he took at every draught.

Ans. He took 27 gallons at every draught; therefore as there were 81 gallons of liquor in the vessel before every draught was made, he took $\frac{27}{81}$ or $\frac{1}{3}$ part of all the liquor in the vessel at every draught: but a mixture of wine and water, whenever it is made, may be supposed to be uniform; therefore in taking a third part of all the liquor in the vessel, he must not only take a third part of all the pure water in the vessel, but also a third part of all the pure wine; with this difference however, that the water was again recruited by filling up the vessel, whereas the wine was not: since then $\frac{1}{3}$ part of the wine was taken at every draught, it follows, that whatever quantity of pure wine was left in the vessel before any draught was made, there would only remain $\frac{2}{3}$ of that quantity afterwards; but the quantity of wine at the beginning was 81 gallons, *ex hypothesi*; therefore $\frac{2}{3}$ of 81, or 54 gallons must be left after the first draught; $\frac{2}{3}$ of 54, or 36 gallons after the second draught; $\frac{2}{3}$ of 36, or 24 after the third draught; and $\frac{2}{3}$ of 24, or 16 after the fourth draught, as the problem requires.

SOLUTION.

As there was the same quantity of liquor in the vessel before every draught was made, to wit, 81 gallons, and as there was taken the same quantity at every draught, it follows, that the same quantity must be left after every draught; for this put x; then will $\frac{x}{81}$ shew what part or parts of all the liquor in the vessel was left after every draught, and consequently of the wine; whence the quantity of wine left in the vessel after every draught may be found thus; there were 81 gallons at the beginning, therefore $81 \times \frac{x}{81}$, or x, must be left after the first draught; $x \times \frac{x}{81}$, or $\frac{x^2}{81}$ must be left after the second draught; $\frac{x^2}{81} \times \frac{x}{81}$, or $\frac{x^3}{6561}$ after the third; and $\frac{x^3}{6561} \times \frac{x}{81}$, or $\frac{x^4}{531441}$ after the fourth; but according to the problem, there were left after the fourth draught 16 gallons; therefore $\frac{x^4}{531441} = 16$; therefore $x^4 = 8503056$; therefore $x^2 = \sqrt{8503056}$

$\sqrt{8503056} = 2916$; therefore x, or the number of gallons left in the vessel after every draught $= \sqrt{2916} = 54$; therefore $81 - x$, or the quantity taken at every draught must be 27, as above.

PROBLEM 39.

64. *It is required to divide the number* 90 *into two such parts, that one part may be to the other as* 2 *to* 3.

Ans. The numbers are 36 and 54: for in the first place, $36 + 54 = 90$; and in the next place, if both 36 and 54 be divided by 18, the quotients will be 2 and 3; whence I infer, that 36 is to 54 as 2 to 3; for a common division by the same number cannot alter the proportion of the numbers divided; and therefore if, after this common division, the quotients be to one another as 2 to 3, the dividends must be also in the same proportion.

SOLUTION.

Put x for the less part, and $90 - x$ for the other; then will x be to $90 - x$ as 2 to 3, by the supposition; but by art. 15, whenever there are four proportionals, the product of the extremes will be equal to the product of the middle terms: here the extremes are x and 3, whose product is $3x$; and the middle terms are $\overline{90 - x}$ and 2, whose product is $180 - 2x$; therefore $3x = 180 - 2x$; therefore $5x = 180$; and x, the less part $= 36$; and $90 - x$, the greater $= 54$, as above.

PROBLEM 40.

65. *Two weights, one of* 5 *pounds, the other of* 7, *are suspended at the extremities of a very small rod* 36 *inches long: I demand the point in the rod, where these two weights will be in* æquilibrio; *that is, I demand the distance of this point from either extremity of the rod.*

Ans. The distance of the point of *æquilibrium* from the greater weight was 15 inches, and consequently $36 - 15$, or 21 inches from the less: for if the weights had been equal, the point of *æquilibrium* would have been exactly in the middle of the rod; but as one weight is less than the other, it will be proportionably nearer the greater weight; and thus 15 is to 21 as $\frac{15}{3}$ is to $\frac{21}{3}$, that is, as 5 to 7.

SOLUTION.

Put x for the distance of the point of *æquilibrium* from the greater weight, and $36 - x$ for it's distance from the less; then will x be to $36 - x$ as 5 to 7; whence by multiplying extremes and means, according

ing to art. 15, we have this equation, $7x=180-5x$; whence $12x=180$; and x, the diſtance of the point of *æquilibrium* from the greater weight $=15$, and $36-x$, it's diſtance from the leſs $=21$, as above.

PROBLEM 41.

66. *What number is that, which being ſeverally added to 36 and 52, will make the former ſum to the latter as 3 to 4?*

Anſ. The number is 12: for $36+12$ is to $52+12$, as 48 is to 64, as $\frac{48}{16}$ is to $\frac{64}{16}$, as 3 to 4.

SOLUTION.

Put x for the number ſought, and you will have this proportion; $36+x$ is to $52+x$ as 3 to 4. Whence by multiplying extremes and means you will have $144+4x=156+3x$; therefore $144+x=156$; therefore x, the number ſought $=12$, as above.

PROBLEM 42.

67. *A bookbinder ſells me two paper books, one containing 48 ſheets for 3 ſhillings and 4 pence, and another containing 75 ſheets for 4 ſhillings and 10 pence, both bound at the ſame price, and both of the ſame ſort of paper: I demand what he allows himſelf for binding.*

Anſ. He reckoned 8 pence for binding; ſo that the price of the paper of the firſt book was 32 pence, and the price of the paper of the latter 50 pence: now if this anſwer be juſt, the two prices ought to bear the ſame proportion to one another as the two quantities of paper; and ſo we ſhall find them: for 32 pence are to 50 pence as $\frac{32}{2}$ are to $\frac{50}{2}$, that is, as 16 to 25; and 48 ſheets are to 75 ſheets as $\frac{48}{3}$ are to $\frac{75}{3}$, that is alſo, as 16 to 25.

SOLUTION.

Put x for the number of pence reckoned for binding; then we ſhall have $40-x$ for the price of the paper in the firſt book, and $58-x$ for the price of the paper in the ſecond book; and $40-x$ will be to $58-x$ as 48 to 75; multiply extremes and means, and you will have this equation, $2784-48x=3000-75x$; therefore $2784+27x=3000$; therefore $27x=216$; and x, the number of pence reckoned for binding $=8$, as above.

PROBLEM 43.

68. *What number is that, which being ſeverally added to 15, 27, and 45, will give three numbers in continual proportion?*

N. B.

N. B. Three numbers are ſaid to be in continual proportion, when the firſt is to the ſecond as the ſecond is to the third.

Anſ. The number ſought is 9: for $15+9=24$; and $27+9=36$; and $45+9=54$; and 24 is to 36 as 36 is to 54: for 24 is to 36 as $\frac{24}{12}$ is to $\frac{36}{12}$, that is, as 2 to 3; and 36 is to 54 as $\frac{36}{18}$ is to $\frac{54}{18}$, that is alſo, as 2 to 3.

SOLUTION.

Put x for the number ſought; then we ſhall have this proportion, $x+15$ is to $x+27$, as $x+27$ is to $x+45$; where the two middle terms are $x+27$ and $x+27$: multiply extremes and means, and you will have this equation, $xx+60x+675=xx+54x+729$; therefore $60x+675=54x+729$; therefore $6x+675=729$; therefore $6x=54$; and x, the number ſought $=9$, as above.

PROBLEM 44.

69. *One places a certain number of rods upright in a ſtreight line, at equal diſtances one from another, the vacancies being no more than ſufficient to contain two rods apiece; but finding that by this means, his line would not reach above* 125 *inches, he extended it to* 208 *inches by opening the vacancies juſt as wide again as before: What was his number of rods?*

Anſ. The number of rods was 84, and conſequently the number of intervals 83: for if two rods admit but of one interval, three rods of 2, &c, 84 rods will admit of 83 intervals, which intervals if they were to be filled, would take up 83×2, or 166 rods; therefore if the firſt line had been full, it would have taken up $84+166$, or 250 rods: again, the number of rods ſufficient to fill the vacancies of the ſecond line was 83×4, or 332; therefore if the ſecond line had been full, it would have taken up $84+332$, or 416 rods: now if this anſwer be juſt, the lengths of theſe two lines ought to have the ſame proportion to one another, as have the number of rods they would have taken up had they been full; and ſo we ſhall find them: for 125 inches are to 208 inches as 125×2 to 208×2, that is, as 250 rods to 416 rods, as was to be demonſtrated.

SOLUTION.

Put x for the number of rods; then will the number of vacancies be $x-1$; and the number of rods ſufficient to fill the vacancies of the firſt line, $2x-2$;, and the number of rods the firſt line would have taken, had it been full, $3x-2$: again, the number of rods ſufficient to fill the vacancies of the ſecond line will be $4x-4$; and therefore the number of

of rods the ſecond line would have taken, had it been full, will be $5x-4$; whence we ſhall have this proportion, $3x-2$ is to $5x-4$ as 125 is to 208; and by multiplying extremes and means, this equation, $624x-416=625x-500$; therefore $x-500=-416$; therefore x, the number of rods $=84$, as above.

Of the method of reſolving problems wherein more unknown quantities than one are concerned, and repreſented by different letters.

70. Hitherto we have uſed but one ſingle letter in every problem for ſome one unknown quantity in it; and if there were more, the reſt received their names from the conditions of the problem: but in caſes of a more complicated nature, where many unknown quantities are linked and entangled in one another, this method will be found very difficult; and therefore in ſuch caſes, the Algebriſt is allowed to uſe as many different letters as he has unknown quantities, provided he finds out as many independent equations for diſcovering their values; ſee art. 92: for though in every equation wherein more unknown quantities than one are concerned, they hinder one another from being found out, yet if as many fundamental equations at firſt be given, as there are unknown quantities, it will not be difficult in many caſes, from theſe to derive others that are more ſimple, till at laſt you come to an equation wherein but one only unknown quantity is concerned, in which caſe all the reſt are ſaid to be exterminated.

Whenever two or more equations are propoſed, involving as many unknown quantities, theſe equations muſt firſt be prepared by freeing them from fractions wherever there are any, and by ordering every particular equation ſo, that all the unknown quantities may poſſeſs one ſide of the equation, and ſuch as are known the other; or elſe, that all the quantities may poſſeſs one ſide of the equation, and a cypher the other: it will be alſo convenient, that in every particular equation, the unknown quantities be placed in the ſame order.

In laying down rules for exterminating unknown quantities, I ſhall begin with the ſimpleſt caſe firſt, which is that of two equations and two unknown quantities; and when I have given as many examples as ſhall be thought proper in this caſe, I ſhall then proceed to others where more unknown quantities are to be exterminated.

But here I muſt not forget to advertiſe the reader, that as I am now treating of ſimple equations, and problems producing ſuch equations, I ſhall not meddle with any caſes of extermination which lead to equations

of higher forms: when I come to treat of quadratic equations, I may then perhaps add ſomething further upon this ſubject; but to undertake to explain all the various methods of exterminating unknown quantities would be an endleſs taſk, and a moſt intolerably laborious and tedious one both to the writer and the reader, whom I cannot yet ſuppoſe to be ſo far gone in Analytics, as to be willing to purchaſe this ſort of knowledge at any rate.

Let then x and y be two unknown quantities to be found out by the help of the two following equations, $4x - 5y = 2$, and $6x - 7y = 4$; or the queſtion may be ſtated thus; if $4x - 5y = 2$, and $6x - 7y = 4$, what are x and y? now as theſe equations want no preparation, put them down one under another; then upon a bye piece of paper multiply the firſt equation ($4x - 5y = 2$) by 6 the coefficient of x in the ſecond equation, and the product will give this equation, $24x - 30y = 12$; again, multiply the ſecond equation ($6x - 7y = 4$) by 4, the coefficient of x in the firſt equation, and the product gives $24x - 28y = 16$; ſubtract now either of theſe two laſt equations from the other, and x will be exterminated: I chooſe in the preſent caſe to ſubtract the former equation from the latter, that the coefficient of y after ſubtraction may affirmative, thus;

$$\begin{array}{r} 24x - 28y = 16 \\ 24x - 30y = 12 \\ \hline * + 2y = 4. \end{array}$$

From this ſubtraction you have the following equation, $2y = 4$, which put down under the two firſt equations to make a third; then reſolve this third equation $2y = 4$, and you will have $y = 2$, which put down under the reſt for a fourth equation.

Having thus found the value of $y = 2$, put this value inſtead of y in the more ſimple of the two firſt equations, ſuppoſe in the equation $4x - 5y = 2$, and you will have $4x - 10 = 2$; whence $4x = 12$, and $x = 3$, which put down for a fifth equation, and the work is done; for x is now found equal to 3, and y equal to 2, and theſe numbers three and two being ſubſtituted for x and y reſpectively, will anſwer both the conditions of the queſtion, that is, you will have $4x - 5y = 12 - 10 = 2$, and $6x - 7y = 18 - 14 = 4$.

1ſt Equ.	$4x - 5y = 2.$
2d,	$6x - 7y = 4.$
3d,	$* + 2y = 4.$
4th,	$* \quad y = 2.$
5th,	$x \quad * = 3.$

 The

The coefficients of x, the quantity to be exterminated in the two firſt equations were 4 and 6: now as theſe numbers admit of a common divisor without any remainder, namely 2, divide them both by 2, and the quotients will be 2 and 3; uſe now theſe numbers 2 and 3 inſtead of 4 and 6, and the operation as well as the equation reſulting from it, will become more ſimple: for the firſt equation multiplied by 3 inſtead of 6, gives $12x - 15y = 6$; and the ſecond equation multiplied by 2 inſtead of 4, gives $12x - 14y = 8$; and the difference of theſe two equations is $y = 2$.

Another way of exterminating the unknown quantity x, is as follows: find out the value of x in reſpect of y, in the more ſimple of the two firſt equations; then ſubſtitute this value inſtead of x in the other equation, you will have an equation, wherein y alone is concerned: thus in the foregoing example, the firſt equation was $4x - 5y = 2$, therefore $4x = 5y + 2$, and $x = \frac{5y+2}{4}$; ſubſtitute now this value $\left(\frac{5y+2}{4}\right)$ inſtead of x in the ſecond equation, $6x - 7y = 4$, by making $6x = \frac{30y+12}{4}$, and you will have this equation, $\frac{30y+12}{4} - 7y = 4$; therefore $30y + 12 - 28y = 16$; therefore $2y + 12 = 16$; whence $2y = 4$, and $y = 2$; and x, or $\frac{5y+2}{4} = 3$, as before.

N. B. 1*ſt*, What has here been ſaid concerning the extermination of the quantity x, may as well be applied to the other quantity y, except that it's coefficients 5 and 7 will not admit of a common diviſor, as did the numbers 4 and 6.

2*dly*, Of the two different ways of extermination here laid down, ſometimes one will be found more expeditious, and ſometimes the other, as will appear by the following problems.

3*dly*, In the caſe of two unknown quantities, if the value of either of them can be had in integral terms in both equations, equate the two values one to the other, and you will have the other unknown quantity, by means whereof the firſt will alſo be known; and this makes a third way of extermination, whereof there are ſo many examples in the following problems, that nothing more needs here to be ſaid of it.

Whenever two quantities, as x and y, are multiplied together to produce a third xy, the two multiplicants x and y are called factors, or efficients, in which caſe, each is ſaid to be the others coefficient: thus in the quantity xy, x is ſaid to be the coefficient of y, and y the coefficient of x; therefore if in any quantity wherein x is concerned as an efficient,

ent, it's coefficient be desired; divide that quantity by x, and the quotient will be the coefficient: thus if the quantity $12x - yx$ be divided by x, the quotient is $12 - y$; therefore in the quantity $12x - yx$, the coefficient of x is $12 - y$

ADVERTISEMENT.

The reader must now no longer expect to have all simple equations resolved to his hand, as hitherto has been done. If after sixteen examples of simple equations resolved, and the solution of forty four Algebraic problems, he be still at a loss how to reduce a simple equation, it must proceed from a weakness that either admits of no cure, or deserves none.

PROBLEM 45.

71. *What two numbers are those, the product of whose multiplication is* 144, *and the quotient of the greater divided by the less is* 16?

SOLUTION.

Put x for the greater number, and y for the less; and the question when abstracted from words will stand thus: if $xy = 144$, and $\frac{x}{y} = 16$, what are x and y?

The first of these equations wants no preparation, and therefore may be put down thus;

Equ. 1st, $xy \quad * = 144$.

The second equation, when prepared according to the last art. will stand thus; Equ. 2d, $x - 16y = 0$.

Multiply the first equation by 1, the supposed coefficient of x in the second, and the equation not being altered by such a multiplication, will be $xy * = 144$; multiply also the second equation by y, which according to the foregoing art. is the coefficient of x in the first, and you will have $xy - 16yy = 0$; subtract this latter product from the former, and you will have,

Equ. 3d, $* \ 16yy = 144$; whence

Equ. 4th, $* \quad y = 3$.

Substitute now 3 instead of y, or $3x$ instead of xy in the first equation, and you will have $3x = 144$, and consequently,

Equ. 5th, $x \quad * = 48$.

So that the numbers at last are found to be 48 and 3; and they will answer the conditions of the question: for $48 \times 3 = 144$, and $\frac{48}{3} = 16$.

Equ. 1ſt, $xy \quad * = 144$.
2d, $x - 16y = 0$.
3d, $* \quad 16yy = 144$.
4th, $* \quad y = 3$.
5th, $x \quad * = 48$.

Another ſolution of the foregoing problem, from the laſt article.

Having found from the ſecond equation that $x = 16y$, put $16y$ for x, or $16yy$ for xy in the firſt equation, and you will have $16yy = 144$; whence y and x may be found as before.

PROBLEM 46.

72. *It is required to find two numbers with the following properties, to wit, that the firſt with half the ſecond may make* 20; *and moreover, that the ſecond with a third part of the firſt may make* 20.

SOLUTION.

Put x for the firſt number, and y for the ſecond; and the fundamental equations will be $x + \frac{y}{2} = 20$, and $y + \frac{x}{3} = 20$; which being prepared according to art. 70, will ſtand thus;

Equ. 1ſt, $2x + y = 40$.
Equ. 2d, $x + 3y = 60$.

Subtract the firſt equation from twice the ſecond, and you will have

Equ. 3d, $* \quad 5y = 80$; whence
Equ. 4th, $* \quad y = 16$.

Put 16 inſtead of y in the firſt equation, and you will have $2x + 16 = 40$, whence

Equ. 5th, $x \quad * = 12$.

Therefore the numbers ſought are 12 and 16, and not 16 and 12, though 16 was found firſt; becauſe $x = 12$ was put for the firſt number. That theſe numbers will anſwer the conditions of the queſtion is plain: for $12 + \frac{16}{2}$ or $12 + 8 = 20$; and $16 + \frac{12}{3}$, or $16 + 4 = 20$.

Another ſolution from art. 70.

Having found from the ſecond equation that $x = 60 - 3y$, put $60 - 3y$ for x, or $120 - 6y$ for $2x$ in the firſt equation, and you will have $120 - 6y + y = 40$; whence $y = 16$, as before.

PROB-

PROBLEM 47.

73. *One exchanges 6 french crowns and two french dollars for 45 shillings; and at another time 9 crowns and 5 dollars of the same coin for 76 shillings: I demand the distinct values of a crown and of a dollar.*

SOLUTION.

Put x and y for the number of shillings a crown and a dollar are respectively worth, and the equations will stand thus;

Equ. 1st, $6x+2y=45$.
Equ. 2d, $9x+5y=76$.

Subtract 3 times the first equation from twice the second, and you will have
Equ. 3d, $* \quad 4y=17$; whence
Equ. 4th, $* \quad y=4\frac{1}{4}$ shillings;

that is, 4 shillings and 3 pence; put now $4\frac{1}{4}$ for y, or $8\frac{1}{2}$ for $2y$ in the first equation, and you will have $6x+8\frac{1}{2}=45$, and $6x=36\frac{1}{2}$, and
Equ. 5th, $x \quad * =6\frac{1}{12}$;

that is, $6\frac{1}{12}$ shillings, or 6 shillings and a penny; therefore the value of a crown was 6 shillings and a penny, and that of a dollar 4 shillings and 3 pence; and these values will answer the conditions of the question; for at this rate, 6 crowns will amount to 36 shillings and 6 pence, 2 dollars to 8 shillings and 6 pence, and the whole to 45 shillings; moreover, 9 crowns will amount to 54 shillings and 9 pence, 5 dollars to 21 shillings and 3 pence, and the whole sum to 76 shillings.

PROBLEM 48.

74. *It is required to find two such numbers, that half the first together with a third part of the second may make 32; and moreover, that a fourth part of the first together with a fifth part of the second may make 18.*

SOLUTION.

Put x and y for the two numbers, and the fundamental equations will be $\frac{x}{2}+\frac{y}{3}=32$, and $\frac{x}{4}+\frac{y}{5}=18$; which equations when duly prepared, will stand thus;

Equ. 1st, $3x+2y=192$.
Equ. 2d, $5x+4y=360$.

Subtract 5 times the first equation from 3 times the second, and you will have

have Equ. 3d, $* \quad 2y = 120$; whence

Equ. 4th, $* \quad y = 60$;

whence, and from the firſt equation, you will have $3x + 2y$, or $3x + 120 = 192$, which gives

Equ. 5th, $x \quad * = 24$.

So the numbers are 24 and 60; and they will anſwer the conditions of the queſtion: for $\frac{24}{2} + \frac{60}{3}$, that is, $12 + 20 = 32$; and moreover, $\frac{24}{4} + \frac{60}{5}$, that is, $6 + 12 = 18$.

PROBLEM 49.

75. *Two perſons* A *and* B *were talking of their ages; ſays* A *to* B, 7 *years agoe I was juſt three times as old as you were, and* 7 *years hence I ſhall be juſt twice as old as you will be: I demand their preſent ages.*

SOLUTION.

Let a and b repreſent the preſent ages of A and B reſpectively; then their ages 7 years agoe were $a - 7$ and $b - 7$, and their ages 7 years hence will be $a + 7$ and $b + 7$; whence, and from the conditions of the problem, may be derived the two following fundamental equations;

$$a - 7 = \overline{b - 7} \times 3 = 3b - 21, \text{ and}$$

$$a + 7 = \overline{b + 7} \times 2 = 2b + 14.$$

From the former of theſe two equations, to wit, $a - 7 = 3b - 21$, we have $a = 3b - 14$; from the ſecond equation, to wit, $a + 7 = 2b + 14$, we have $a = 2b + 7$; therefore $3b - 14 = 2b + 7$, ſince both are equal to a; whence $b = 21$, and $2b + 7$, or $a = 49$.

A therefore was 49 years old, and B 21 years old; which is true: for then, 7 years before, A's age would be 42, and B's 14; and 42 is three times 14: on the other hand, 7 years after, A's age would be 56, and B's 28; and 56 is twice 28.

PROBLEM 50.

76. *A jocky has two horſes,* A *and* B, *whoſe values are ſought; he has alſo two ſaddles, one valued at* 12 *pounds, the other at* 2: *now if he ſets the better ſaddle upon* A *and the worſe ſaddle upon* B, A *will then be worth twice as much as* B; *but on the other hand, if he ſets the better ſaddle upon* B, *and the worſe ſaddle upon* A, B *will then be worth three times as much as* A: *I demand the values of the horſes.*

SOLUTION.

Let a and b repreſent the prices of the two horſes A and B reſpectively in pounds; then if the better ſaddle be ſet upon A, and the worſe upon

on B, A will be worth $a+12$, and B will be worth $b+2$, and the first fundamental equation will be $a+12=\overline{b+2}\times 2=2b+4$; on the other hand, if the better saddle be set upon B, and the worse upon A, then B will be worth $b+12$, and A will be worth $a+2$, and the second fundamental equation will be $b+12=\overline{a+2}\times 3=3a+6$: in the first fundamental equation, where $a+12=2b+4$, we have $a=2b-8$; substitute therefore $2b-8$ instead of a, or rather $6b-24$ instead of $3a$, in the second fundamental equation, (which is $3a+6=b+12$) and you will have $6b-24+6$, that is, $6b-18=b+12$; whence $b=6$, and $2b-8$, or $a=4$: A then was valued at 4 pounds, and B at 6, and they will answer the conditions of the question, as any one may easily try.

PROBLEM 51.

77. *There is a certain fraction, which if an unit be added to the numerator, will be equal to $\frac{1}{3}$; but if on the contrary an unit be added to the denominator, the fraction will then be equivalent to $\frac{1}{4}$: I demand the numerator and denominator of the fraction.*

SOLUTION.

Call the fraction $\frac{x}{y}$, and you will have these two fundamental equations, $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$: the former of these equations when reduced, gives $y=3x+3$, and the latter gives $y=4x-1$; therefore $4x-1=3x+3$, because both are equal to y; whence x the numerator of the fraction is 4; and $3x+3$, or y, the denominator is 15; and the fraction itself is, $\frac{4}{15}$; which if an unit be added to the numerator, will be $\frac{5}{15}$, or $\frac{1}{3}$; but if an unit be added to the denominator, it will be $\frac{4}{16}$, or $\frac{1}{4}$.

PROBLEM 52.

78. *There is a certain fishing rod consisting of two parts, whereof the upper part is to the lower as 5 to 7; and moreover 9 times the upper part together with 13 times the lower, is equal to 11 times the whole rod and 36 inches over: I demand the length of the two parts.*

SOLUTION.

Put x for the length of the upper part in inches, and y for the lower; then will $x+y$ be the length of the whole rod; and since x is to y as 5 to 7 *ex hypothesi*, by multiplying extremes and means according to art.

15, you will have $7x=5y$ for a fundamental equation: again, as 9 times the upper part, together with 13 times the lower, is equal to 11 times the whole rod, and 36 inches over, you have $9x+13y=11x+11y+36$ for a second fundamental equation: the latter of these two equations gives $x=y-18$, and consequently $7x=7y-126$; substitute this value instead of $7x$, in the first fundamental equation, where $7x=5y$, and you will have $7y-126=5y$; whence $y=63$; and $y-18$, or $x=45$.

The upper part therefore was 45 inches, and the lower 63, as will appear upon tryal.

PROBLEM 53.

79. *One lays out 2 shillings and sixpence in apples and pears, buying his apples at four, and his pears at five a penny; and afterwards accommodates his neighbour with half his apples and one third part of his pears for thirteenpence, which was the price he bought them at: I demand how many he bought of each sort.*

SOLUTION.

Put x for the number of apples, and y for the number of pears; then if 4 apples cost one penny, x will cost $\frac{x}{4}$ pence; and for the same reason y will cost $\frac{y}{5}$ pence, and you will have $\frac{x}{4}+\frac{y}{5}=30$ for a first fundamental equation: again, the price of $\frac{x}{2}$, half of his apples will be $\frac{x}{8}$; and the price of $\frac{y}{3}$, a third part of his pears will be $\frac{y}{15}$; and you will have $\frac{x}{8}+\frac{y}{15}=13$ for a second fundamental equation. Hence

Equ. 1st, $5x+4y=600$.
Equ. 2d, $15x+8y=1560$.

Subtract the second equation from three times the first, according to art. 70, and you will have

Equ. 3d, $*\ \ 4y=240$; whence
Equ. 4th, $*\ \ y=60$.

Substitute now 60 for y, that is, 240 for $4y$ in the first equation $5x+4y=600$, and you will have $5x+240=600$; whence

Equ. 5th, $x\ \ *=72$.

Therefore the number of apples was 72, and the number of pears 60, as will appear upon tryal.

PROB-

PROBLEM 54.

80. *A greyhound spying a hare at the distance of fifty of his own leaps from him, pursues her with full speed, making three leaps for every four of the hare's; and moreover passing over as much ground in two leaps, as the hare did in three: I demand how many leaps each made during the whole course.*

SOLUTION.

The number of the dog's leaps during the whole course, x;
The number of the hare's leaps in the same time, y:
Therefore while the dog makes x leaps, the hare makes y; but according to the problem, while the dog made three leaps, the hare made four; therefore x is to y as 3 to 4; whence by multiplying extremes and means we have $4x = 3y$: again, from the hare's form to the end of the course, the dog made $x - 50$ leaps, and passed over as much ground as the hare did in all her's y; but according to the problem, the dog passed over as much ground in two leaps as the hare did in three; therefore $x - 50$ is to y as 2 to 3; whence again by multiplying extremes and means we have $3x - 150 = 2y$: the rest of the solution is as follows;

Equ. 1st, $4x - 3y = 0$.
Equ. 2d, $3x - 2y = 150$.

Subtract three times the first equation from four times the second, and you will have

Equ. 3d, $*\quad y = 600$.

Put 600 for y in the first equation, and you will have $4x - 3y$, that is, $4x - 1800 = 0$; whence

Equ. 4th, $x \quad * = 450$.

Therefore the dog made 450 leaps, and the hare 600 during the whole course; and 450 is to 600 as $\frac{450}{150}$ is to $\frac{600}{150}$, that is, as 3 to 4: again, from the hare's form to the end of the course the dog made 400 leaps; and 400 is to 600 as 4 to 6, or as 2 to 3.

PROBLEM 55.

81. *It is required to find two numbers* x *and* y *of such a nature, that if both be multiplied by* 18, *the first product will be a square, and the second will be the side or root of that square; but if both be multiplied by* 3, *the first product will be a cube, and the second the side of that cube.*

SOLUTION.

If x and y be both multiplied by 18, the products will be $18x$ and $18y$, whereof the former is to be a square, and the latter the side of that square, that is, in other words, the former product is to be equal to the square of the latter; and so we have $18x = 324yy$, and $x = 18yy$: again, if x and y be multiplied by 3, the products will be $3x$ and $3y$, whereof the former is to be equal to the cube of the latter; and so we have $3x = 27y^3$, and $x = 9y^3$; therefore $9y^3 = 18y^2$, because both are found equal to x; therefore $y = 2$, and $9y^3$, or $x = 72$; so that the numbers sought are 72 and 2, which will answer the conditions; for if both be multiplied by 18, the products will be 1296 and 36, whereof the former is a square, and the latter it's side; but if the numbers 72 and 2, be both multiplied by 3, the products will be 216 and 6, whereof the former is a cube, and the latter it's side.

PROBLEM 56.

82. *There is a certain floor in form of a rectangle or long square, whose dimensions are such, that if it had been two feet broader, and three feet longer, it would have been sixty four square feet larger; but if on the other hand, it had been three feet broader and two feet longer, it would then have been sixty eight square feet larger: I demand the length and breadth of the floor.*

SOLUTION.

Put x for the breadth, and y for the length of the floor; then will it's area, or number of square feet it contains be $x \times y$ or xy, as every one knows; but if it had been 2 feet broader and 3 feet longer, it's area would then have been $\overline{x+2} \times \overline{y+3}$, that is, $xy + 3x + 2y + 6$; and as by supposition, this area is to exceed the true one by 64 square feet, we have the following equation, $xy + 64 = xy + 3x + 2y + 6$: again, if this floor had been 3 feet broader and 2 feet longer, its area would then have been $\overline{x+3} \times \overline{y+2}$, that is, $xy + 2x + 3y + 6$; and this last area is to exceed the true one by 68 square feet; whence we have this equation, $xy + 68 = xy + 2x + 3y + 6$; whence

Equ. 1st, $3x + 2y = 58$.
Equ. 2d, $2x + 3y = 62$.

Subtract twice the first equation from three times the second, and you will have

Equ. 3d, * $5y = 70$; and
Equ. 4th, * $y = 14$.

Sub-

Substitute now 14 for y, or 28 for $2y$ in the first equation $3x+2y=58$, and you will have $3x+28=58$, whence

Equ. 5th, $x \quad * = 10$.

The floor therefore was 10 feet broad, and 14 feet long; and its area 140 square feet, which answers the conditions of the question: for then, if it had been 2 feet broader and 3 feet longer, its area would have been $12\times17=204=140+64$; but if it had been 3 feet broader and 2 feet longer, its area would have been $13\times16=208=140+68$.

PROBLEM 57.

83. *A certain company at a tavern found, when they came to pay their reckoning, that if they had been three more in company to the same reckoning, they might have paid one shilling apiece less than they did; and that, had they been two fewer in company, they must have paid one shilling apiece more than they did: I demand the number of persons, and their* quota.

SOLUTION.

Put x for the number of persons, and y for the number of shillings every one actually paid; now if 4 persons are to pay 5 shillings apiece, the whole reckoning must be 4×5 or 20 shillings; therefore if x persons are to pay y shillings apiece, the whole reckoning must be $y\times x$ or xy shillings: this being laid down, suppose them now to be three more in company; then will the number of persons be $x+3$; and to find what every particular person ought to pay in this case, the whole reckoning xy, must be divided by $x+3$, the number of persons, and the quotient $\frac{xy}{x+3}$ will be every one's particular reckoning; but according to the problem, every ones particular reckoning in this case would have been one shilling less than it actually was, that is, $y-1$; therefore $\frac{xy}{x+3}= y-1$; in like manner the second condition of the problem furnishes this equation, $\frac{xy}{x-2}=y+1$: the first of these equations, to wit, $\frac{xy}{x+3}= y-1$, being reduced, gives $x=3y-3$; and the second equation, to wit, $\frac{xy}{x-2}=y+1$ being reduced gives $x=2y+2$; therefore $3y-3=2y+2$, and $y=5$; whence $2y+2$, or $x=12$.

So there were 12 persons in company, their reckoning 5 shillings apiece, and their whole reckoning 3 pounds, or 60 shillings; which answers the conditions of the question: for $\frac{60}{15}=4$, and $\frac{60}{10}=6$.

PROBLEM 58.

84. *A vintner has two sorts of wines, a better sort and a worse, whose values are such, that if he mixes them in the proportion of two to three, that is, after the rate of two quarts of the better sort to three of the worse, the mixture will be worth one and twenty pence a quart; but if he mixes them in the proportion of seven to eight, that is, after the rate of seven quarts of the better sort to eight of the worse, the mixture will then be worth two and twenty pence the quart: I demand the price of a quart of each sort.*

SOLUTION.

Put x for the price in pence of a quart of the better sort, and y for the price of a quart of the worse in pence: then if he mixes 2 quarts of the better sort with 3 of the worse, the mixture will be worth $2x+3y$; but as there were 5 quarts of this mixture, valued at 21 pence a quart, the whole mixture must be worth 105 pence; whence we have this equation, $2x+3y=105$: in like manner the other condition of the problem furnishes this equation, $7x+8y=22\times15=330$; which equations are solved as follows;

Equ. 1st,	$2x+3y=105.$	Equ. 4th,	$*\quad y=15.$
Equ. 2d,	$7x+8y=330.$	Equ. 5th,	$x\quad *\ =30.$
Equ. 3d,	$*\quad 5y=75.$		

So the better sort of wine was worth 30 pence a quart, and the worse 15; which answers the conditions of the problem: for at this rate, two quarts of the better sort will be worth 60 pence, three of the worse 45 pence, the whole five quarts 105 pence, and a single quart 21 pence; again, seven quarts of the better sort will be worth 210 pence, eight of the worse 120 pence, the whole fifteen quarts 330 pence, and a single quart 22 pence.

PROBLEM 59.

85. *There is a certain number consisting of two places, which is equal to four times the sum of it's digits; and if to the number be added 18, the digits will be inverted: I demand the number.*

N. B. By the digits of any number are meant the figures that represent it, without any regard had to their places; thus the digits of the number 36 are 3 and 6, whose sum is 9; whence it appears, that the number 36 has the first property described in the problem, to wit, that it is equal to 4 times 9, or 4 times the sum of it's digits; but it is not the true number sought, because it has not the second property; for if it had, then 36 and 18 together would make 63, a number with the former digits inverted; whereas they only make 54.

SOLU-

SOLUTION.

Put x for the digit in the ten's place, and y for the digit in the place of units, and the ſum of the digits will be $x+y$; but how to read ſuch a number as this, that hath x in the ten's place and y in the place of units, comes next to be conſidered: now in order to do this, the reader muſt reflect what paſſes in his own mind when he ſees a number of two places, as 36; then he will eaſily ſee, that he makes the 6 in the place of units ſtand for no more than it's value, but the 3 in the place of tens he makes to ſtand not for 3, but for 30, a number 10 times as much; ſo that he reads the number 36, *quaſi* $30+6$; therefore a number that has x in the place of tens, and y in the place of units, ought to be read thus, $10x+y$; and a number with the ſame digits inverted muſt be read thus, $10y+x$: this being well underſtood, the fundamental equations flow eaſily from the conditions of the problem thus; $10x+y=4x+4y$, and $10x+y+18=10y+x$. The former equation reduced gives $y=2x$, and the latter gives $y=x+2$; therefore $2x=x+2$; whence $x=2$, $2x$ or $y=4$, and the number ſought is 24; the ſum of whoſe digits is 6: and that it anſwers the conditions of the problem, is plain; for $24=4$ times 6; and $24+18=42$.

A LEMMA.

86. *If there be a compound quantity conſiſting of any two multiples of* x *and* y, *whereof one is affirmative and the other negative, and if this compound quantity is to be ſubtracted from* x + y; *this ſubtraction may be eaſily made without the work of the pen, thus: change the ſigns of the two parts to be ſubtracted; then encreaſe the coefficient of the affirmative part, and diminiſh the coefficient of the negative part by unity, and you will have the remainder.* Thus if $6x-10y$ is to be ſubtracted from $x+y$, firſt change the ſigns $6x-10y$ and you will have $10y-6x$; then encreaſe the affirmative coefficient 10, and diminiſh the negative coefficient 6 by unity, and you will have $11y-5x$ for a remainder, as may eaſily be ſeen by actually ſubtracting $6x-10y$ from $x+y$: again, if $2y$, that is, $2y-0x$ is to be ſubtracted from $x+y$, the remainder will be $x-y$.

PROBLEM 60.

87. *Two perſons* A *and* B *have each a certain number of counters*; A *gives to* B *as many as* B *has already*; *after which*, B *returns back again to* A *as many as* A *has left*; *then* A *returns again to* B *as many as* B *has left*, *and ſo they go on*; *and after four exchanges thus made, they had each of them ſixteen counters: I demand how many each had at firſt.*

SOLU-

SOLUTION.

Put x for *A*'s number of counters, and y for that of *B*'s; then it is plain that the ſum of both their counters at the beginning will be $x+y$; and if ſo, then it is as plain, that the ſum of both their counters will ever afterwards be $x+y$; becauſe, whatever exchanges were made backwards and forwards, there were no counters loſt between them: therefore if in any caſe an expreſſion for either of their counters be given, it is but ſubtracting that expreſſion from $x+y$, and the remainder will be an expreſſion for the other's counters; whence we have the following ſolution:

A's number of counters, x.
B's, y.
B's after the firſt exchange, $2y$.
A's, $x-y$.
A's after the ſecond exchange, $2x-2y$.
B's, $3y-x$.
B's after the third exchange, $6y-2x$.
A's, $3x-5y$.
A's after the fourth exchange, $6x-10y$.
B's, $11y-5x$.

And thus may the exchanges be carried on with eaſe to any number at pleaſure: but according to the problem, after four exchanges thus made, they had each of them 16 counters; whence we have theſe two equations, $6x-10y=16$, and $11y-5x=16$: the reſolution follows;

Equ. 1ſt,	$3x-5y=8$.		Equ. 4th,	*	$y=11$.
2d,	$5x-11y=-16$.		5th,	x	$*=21$.
3d,	* $8y=88$.				

So that *A* had 21 counters, and B 11; and the account will ſtand thus:
After the firſt exchange, *B* has 22 counters, and *A* has $21-11=10$.
After the ſecond exchange, *A* has 20, and *B* has $22-10=12$.
After the third exchange, *B* has 24, and *A* has $20-12=8$.
After the fourth exchange, *A* has 16, and *B* has $24-8=16$.

Obſervations upon the foregoing problem.

If any one has curioſity enough to enquire by what contrivance in this and many other problems, ſuch *data* can be made choice of as that the anſwers ſhall come out in whole numbers; whereas, were thoſe *data* aſſumed arbitrarily and at random, the anſwers would, generally ſpeaking, come out in fractional numbers; this is effected by the following artifice: ſuppoſe that in this problem, after four exchanges made between *A* and *B*,

I had

I had a mind they ſhould both have the ſame number of counters, but did not know what number to aſſign them at laſt, ſo as that each ſhould have a whole number of counters at firſt; I put down ſome letter, as c, for the number of counters each had left at laſt; then the two fundamental equations will ſtand thus, $6x - 10y = c$, and $11y - 5x = c$; and muſt be reſolved thus:

Equ. 1ſt, $6x - 10y = c$.	Equ. 3d, * $16y = 11c$.
Equ. 2d, $5x - 11y = -c$.	Equ. 4th, * $y = \frac{11c}{16}$.

Therefore by the firſt equation, $6x - 10y$, that is, $6x - \frac{110c}{16} = c$; therefore $96x - 110c = 16c$; therefore $96x = 126c$; therefore

Equ. 5th, x * $= \frac{126c}{96} = \frac{21c}{16}$.

So that, to make each perſon *A* and *B*, have the ſame number c of counters at laſt, *A*'s firſt number muſt be $\frac{21c}{16}$, and *B*'s $\frac{11c}{16}$; whence it follows, that if for the number c, I make choice of any number that can be divided by 16 without any remainder, then both *A* and *B* will neceſſarily have a whole number at firſt; for if $\frac{c}{16}$ be a whole number, then both $\frac{21c}{16}$ and $\frac{11c}{16}$ muſt be ſo too; and this is the reaſon that I made choice of the number 16 itſelf to ſtand for c in this problem; for then I knew, not only that the anſwer would come out in whole numbers, but alſo, that this problem would be the ſimpleſt of its kind, ſince 16 is the leaſt number that can be divided by 16.

This problem may alſo be ſolved by the help of one ſingle letter only, thus: make c now to repreſent, not the number of counters each had at laſt, but the ſum of both their counters at laſt, and conſequently the ſum of both their counters all along, and the lemma for carrying on the exchanges to any number at pleaſure will be this; *If there be a compound quantity conſiſting of any two multiples of* c *and* x, *whereof one is affirmative and the other negative, and if this compound quantity is to be ſubtracted from* c, *the remainder may be obtained, firſt by changing the ſigns of the two parts, and then by encreaſing or diminiſhing the coefficient of* c *by unity, according as it happens after this change to be affirmative or negative.* Thus if $2c - 2x$ is to be ſubtracted from c, the remainder will be $2x - c$; and if $4x - 2c$ is to be ſubtracted from c, the remainder will be $3c - 4x$; whence we have this ſolution of the foregoing problem:

A's

A's number of counters, x.
B's, $c-x$.
B's after the firſt exchange, $2c-2x$.
A's, $2x-c$.
A's after the ſecond exchange, $4x-2c$.
B's, $3c-4x$.
B's after the third exchange, $6c-8x$.
A's, $8x-5c$.
A's after the fourth exchange, $16x-10c$.
B's, $11c-16x$.

But according to the problem, after the fourth exchange each had $\frac{1}{2}c$. therefore x may be found, and will be the ſame whether we make $16x-10c=\frac{c}{2}$, or $11c-16x=\frac{c}{2}$: if we make $16x-10c=\frac{c}{2}$, we ſhall have $32x-20c=c$, and $32x=21c$, and x, or *A*'s number $=\frac{21c}{32}$; whence $c-x$, or *B*'s number $=\frac{11c}{32}$; therefore the ſimpleſt problem of this kind (I mean in the caſe of four exchanges) that will admit of a whole number ſolution, is when c is made equal to 32.

PROBLEM 61.

88. *What two numbers are thoſe, whoſe ſum is twice, and the product of whoſe multiplication is twelve times their difference?*

SOLUTION.

Put x for the greater number, and y for the leſs; then will their difference be $x-y$, their ſum $x+y$, and the product of their multiplication xy or yx; and the equations will be $x+y=2x-2y$, and $yx=12x-12y$; whence

Equ. 1ſt, $x-3y=0$.
Equ. 2d, $12x-yx-12y=0$.

Multiply the firſt equation by $12-y$, which by art. 70, is the coefficient of x in the ſecond, and the product will be $12x-yx-36y+3yy=0$; ſubtract this equation from the ſecond, and you will have

Equ. 3d, $24y-3yy=0$; whence
Equ. 4th, $y=8$; and
Equ. 5th, $x=24$.

And the numbers 24 and 8 will anſwer the conditions.

Otherwiſe thus: by the firſt equation $x=3y$, and $4x=12y$; ſubſtitute $4x$ for $12y$ in the ſecond equation, and you will have $12x-yx-4x=0$; divide by x, and you will have $12-y-4=0$, and $y=8$, and x or $3y=24$, as before.

PROB-

PROBLEM 62.

89. *What two numbers are those, whose difference, sum and product are to each other as are the numbers two, three and five respectively; that is, whose difference is to their sum as two to three, and whose sum is to their product as three to five?*

SOLUTION.

Put x for the greater number, and y for the less; then will their difference be $x-y$, their sum $x+y$, and their product yx; and we shall have these two proportions productive of two equations, 1st, $x-y$ is to $x+y$ as 2 to 3, whence $3x-3y=2x+2y$; 2d, $x+y$ is to yx as 3 to 5, whence $3yx=5x+5y$: the resolution follows;

Equ. 1st, $x-5y=0$.
Equ. 2d, $3yx-5x-5y=0$.

Multiply the first equation by $3y-5$, the coefficient of x in the second, and the product will be $3yx-5x-15yy+25y=0$; subtract this from the second equation, and you will have

Equ. 3d, $15yy-30y=0$; whence
Equ. 4th, $y=2$, and
Equ. 5th, $x=10$.

And the numbers 10 and 2 will answer the conditions of the problem.

Otherwise thus: by the first equation $x=5y$; substitute therefore x instead of $5y$ in the second, and you will have $3yx-5x-x=0$; divide by x and you will have $3y-5-1=0$, and $y=2$, as before.

PROBLEM 63.

90. *It is required to find two numbers such, that if their difference be multiplied into their sum, the product will be five; but if the difference of their squares be multiplied into the sum of their squares, the product will be sixty five.*

SOLUTION.

Put x for the greater number, and y for the less; then will their difference be $x-y$, their sum $x+v$, and the product of their sum and difference multiplied together will be x^2-y^2, by art. 11; then will $x^2-y^2=5$ by the supposition, and $x^2=5+yy$; square both sides, and you will have $x^4=25+10y^2+y^4$: again, the difference of the squares of the two numbers sought is x^2-y^2, and the sum of their squares x^2+y^2, and the product of these two x^4-y^4; therefore $x^4-y^4=65$ by the supposition, and $x^4=65+y^4$; but x^4 was before found equal to

$25+10y^2+y^4$; therefore $25+10y^2+y^4=65+y^4$; whence $y^2=4$, and $y=2$; substitute now 4 for y^2 in the first fundamental equation, which was $x^2-y^2=5$, and you will have $x^2-4=5$, and $x=3$; therefore the numbers sought are 3 and 2, which will answer the conditions.

PROBLEM 64.

91. *There is a certain number consisting of three places, whose digits are in arithmetical progression; if this number be divided by the sum of it's digits, the quotient will be forty eight; and lastly, if from the number be subtracted a hundred ninety eight, the digits will be inverted: I demand the number.*

N. B. Numbers are said to be in arithmetical progression, when they increase or decrease with equal differences: thus the numbers 5, 7 and 9 are said to be in arithmetical progression; as are also 9, 7 and 5. For the better understanding of the following solution of this problem, see problem 59.

SOLUTION.

Here I might put x, y and z for the three digits sought; but as these digits are said to be in arithmetical progression, and because it is esteemed more elegant to work with fewer unknown quantities than with more when it can conveniently be done, I shall put x and $\overline{x+y}$ and $\overline{x+2y}$ for the three digits sought; and since x and $x+y$ and $x+2y$ when added together make $3x+3y$, it is plain that $3x+3y$ will represent the sum of the digits: again, x in the hundred's place stands for $100x$, and $\overline{x+y}$ in the ten's place stands for $10x+10y$, and the quantity $\overline{x+2y}$ in the unit's place stands for itself; and these added together make $111x+12y$; therefore $111x+12y$ will represent the number sought: lastly, when the digits are inverted, $\overline{x+2y}$ in the hundred's place will stand for $100x+200y$, and $x+y$ in the ten's place will stand for $10x+10y$, and x in the place of units will stand for itself; and these added together will make $111x+210y$; therefore $111x+210y$ will represent the number sought with it's digits inverted: these things being premised, the problem furnishes the two following equations, to wit,

$$\frac{111x+12y}{3x+3y}=48, \text{ and}$$

$$111x+12y-198=111x+210y.$$

From the first of these equations, to wit, $\frac{111x+12y}{3x+3y}=48$, we have $144x+144y=111x+12y$; therefore $33x+144y=12y$; therefore $33x+132y=0$; therefore (dividing by 33) $x+4y=0$, and $x=-4y$;

$-4y$.; which is the reſult of the firſt equation: from the other equation, to wit, $111x+12y-198=111x+210y$, we have $210y-12y=-198$, whence $198y=-198$, and $y=-1$; but by the reſult of the former equation, $x=-4y$, that is, $-4\times y$ or $-4\times-1=+4$; therefore x (the firſt digit toward the left hand) $=4$; and ſince y, that is, $+y=-1$, the next digit $x+y$, will be $4-1=3$; and the laſt digit $x+2y$ will be $4-2=2$; therefore the number ſought is 432; the ſum of whoſe digits is 9: for firſt, the digits 4, 3 and 2 are in arithmetical progreſſion; 2dly, $\frac{432}{9}=48$; and 3dly, $432-198=234$.

Though it might have been diſcovered at firſt ſight from the very nature of this laſt problem, that the digits would not increaſe, but decreaſe from left to right, yet I ſuppoſed them to increaſe, only to ſhew, that though in all ſuch caſes we ſhould make wrong ſuppoſitions, yet Algebra would always ſet us right: thus if the gain ariſing from any particular bargain is to be computed, we may put x for it, and ſo proceed as the conditions of the problem require; yet it is not impoſſible but that at the end of the operation x may come out negative; which would ſhew that what we ſuppoſed to be gain, was actually loſs.

Some reflections concerning the conditions of problems.

92. In art. 70 it was ſaid, that the equations by means whereof the values of unknown quantities are diſcovered, ought to be independent one of another; the reaſon of which aſſertion was, that if theſe equations were not independent, they muſt either be conſequent one of another, or contradictory one to another; and all concluſions drawn from them would either be trifling, or abſurd: for if the equations be conſequent one of another, all you arrive to at laſt will be, that ſomething is equal to itſelf; and if the equations be inconſiſtent one with another, you will find at laſt that ſome greater quantity is equal to a leſs: as for example, let the equations be $2x=3y$, and $4x=6y$; then it is plain, that this laſt equation is a conſequence of the former, as being the double of the former; and if you ſhould offer to reſolve theſe two equations by any of the foregoing rules, you would come to no more at laſt than this, that $0=0$: again, let the equations be $2x=3y$, and $4x=6y+7$; here it is plain, that this latter equation contradicts the former; for if $2x$ be equal to $3y$, then $4x$ ought to be equal to $6y$, and not to $6y+7$; and if theſe two equations be reſolved, or be attempted to be reſolved by any of the foregoing rules, you will come at laſt to this abſurdity, that $7=0$; and thus it very often happens that at the latter end of an operation,

equations are discovered to be consequential of, or inconsistent with one another, which at the beginning could not be so easily perceived.

I proceed now to give an example or two of problems wherein more than two unknown quantities are concerned.

PROBLEM 65.

93. *Three persons,* A, B *and* C *were talking of their money; says* A *to* B *and* C, *give me half of your money, and I shall have* d; *says* B *to* A *and* C, *give me a third part of your money, and I shall have* d; *says* C *to* A *and* B, *give me a fourth part of your money, and I shall have* d: *How much money had each?*

N. B. The letter d is here supposed to supply the place of some known quantity, which is left undetermined till the calculation is over.

SOLUTION.

Let a, b and c represent the money of A, B and C respectively, and we shall have these three fundamental equations;

$$a+\frac{b+c}{2}=d;$$

$$b+\frac{a+c}{3}=d; \text{ and}$$

$$c+\frac{a+b}{4}=d.$$

These equations, after due preparations according to art. 70, will stand thus;

Equ. 1st, $2a+b+c=2d.$
Equ. 2d, $a+3b+c=3d.$
Equ. 3d, $a+b+4c=4d.$

Subtract the first equation from twice the second, and you will have

Equ. 4th, $* \quad 5b+c=4d.$

Subtract the third equation from the second, and you will have

Equ. 5th, $* \quad 2b-3c=-d.$

Subtract five times the fifth equation from twice the fourth, and you will have

Equ. 6th, $* \quad * \quad 17c=13d.$

Equ. 7th, $* \quad * \quad c=\frac{13d}{17}.$

Put this value for c in the fourth equation, and you will have $5b+c$, that is, $5b+\frac{13d}{17}=4d$; therefore $85b+13d=68d$; therefore $85b=55d,$

$=55d$, and $b=\frac{55d}{85}=\frac{11d}{17}$; therefore

Equ. 8th, $* \; b \; * = \frac{11d}{17}$.

Put now the two values of b and c already found, instead of b and c in the first equation, and you will have $2a+b+c$, that is, $2a+\frac{11d+13d}{17}$, or $2a+\frac{24d}{17}=2d$; whence $34a+24d=34d$; and $34a=10d$, and $a=\frac{10d}{34}=\frac{5d}{17}$; therefore

Equ. 9th, $a \; * \; * = \frac{5d}{17}$.

So that the numbers are at last found to be $a=\frac{5d}{17}$, $b=\frac{11d}{17}$, and $c=\frac{13d}{17}$; whence it follows, that if any number be put for d, that will admit of the number 17 for a divisor, the quantities a, b and c will come out in whole numbers: as if d be made equal to 17, the quantities a, b and c will be 5, 11 and 13 respectively; and the numbers will answer the conditions of the problem; for $5+\frac{11+13}{2}$, or $5+12=17$; $11+\frac{5+13}{3}$, or $11+6=17$; $13+\frac{5+11}{4}$, or $13+4=17$.

Advertisement. I hope the reader does not need to be told, that the numbers a, b and c must always be understood to be of the same denomination with the number d; as if the number d signifies so many guineas, the numbers a, b and c must also signify guineas; if shillings, shillings; if pence, pence; &c.

Equ. 1st, $2a+b+c=2d$.	Equ. 6th, $* \; * \; 17c=13d$.
2d, $a+3b+c=3d$.	7th, $* \; * \; c=\frac{13d}{17}$.
3d, $a+b+4c=4d$.	8th, $* \; b \; * = \frac{11d}{17}$.
4th, $* \; 5b+c=4d$.	9th, $a \; * \; * = \frac{5d}{17}$.
5th, $* \; 2b-3c=-d$.	

A

A SCHOLIUM.

94. Of the foregoing equations, the firſt, ſecond and third, wherein the quantity a is concerned, may be called equations of the firſt rank; the fourth and fifth, wherein the quantity b is concerned, and out of which the quantity a is excluded, may be called equations of the ſecond rank; the ſixth, wherein c is concerned, and out of which both a and b are excluded, may be called an equation of the third rank; and ſo on, were there ever ſo many unknown quantities.

Whenever the equations of any particular rank are given or found, in order to derive from thence equations of an inferior rank, the Analyſt is at liberty to combine theſe firſt equations by pairs as he pleaſes, provided he does but obſerve theſe two things; firſt, that every equation of the given rank be ſome time or other coupled with ſome other equation of the ſame ſet, ſo as that no equation be left out of the account; ſecondly, that in every particular combination, one of the equations be ſuch as was never made uſe of in any combination before, and the other ſuch as hath been concerned in ſome combination before, excepting the firſt pair. It is not to be denied but that the artiſt may, if he pleaſes, vary ſometimes from this laſt precept; but if he always obſerves it, it will be altogether as well.

PROBLEM 66.

95. *Three perſons,* A, B *and* C *were talking of their money; ſays* A *to* B *and* C, *give me* e *out of your ſtock, and I ſhall have twice as much as you will have left; ſays* B *to* A *and* C, *give me* e *out of your ſtock, and I ſhall have three times as much as you will have left; ſays* C *to* A *and* B, *give me* e *out of your ſtock, and I ſhall have four times as much as you will have left: How much money had each?*

SOLUTION.

Put a, b and c for the money of A, B and C reſpectively, and you will have theſe three fundamental equations;

$$a+e=2b+2c-2e.$$
$$b+e=3a+3c-3e.$$
$$c+e=4a+4b-4e.$$

Which being prepared, ſtand thus;

Equ. 1ſt, $a-2b-2c=-3e.$
Equ. 2d, $3a-b+3c=4e.$
Equ. 3d, $4a+4b-c=5e.$

Subtract

Subtract three times the first equation from the second, and you will have

Equ. 4th, * $5b+9c=13e$.

Subtract four times the first equation from the third, and you will have

Equ. 5th, * $12b+7c=17e$.

Subtract five times the fifth equation from twelve times the fourth, and you will have

Equ. 6th, * * $73c=71e$; whence

Equ. 7th, * * $c=\frac{71e}{73}$.

Put this value instead of c in the fourth equation, and you will have $5b+9c$, that is, $5b+\frac{639e}{73}=13e$; whence $365b+639e=949e$; therefore $365b=310e$, and $b=\frac{310e}{365}=\frac{62e}{73}$; therefore

Equ. 8th, * b * $=\frac{62e}{73}$.

Therefore $b+c=\frac{133e}{73}$, and $2b+2c=\frac{266e}{73}$, and $-2b-2c=-\frac{266e}{73}$; substitute therefore $-\frac{266e}{73}$ instead of $-2b-2c$ in the first equation, and you will have $a-\frac{266e}{73}=-3e$; therefore $73a-266e=-219e$; therefore

Equ. 9th, a * * $=\frac{47e}{73}$.

So that at last, A's money is discovered to be $\frac{47e}{73}$, B's $\frac{62e}{73}$, and C's $\frac{71e}{73}$: make $e=73$, and then A's money will be 47, B's 62, and C's 71, as will easily appear thus; A's stock is 47, B's and C's together make 133; add 73 to 47, and take it from 133, and A will have 120, and B and C will have 60; and $60\times 2=120$: again, B's money is 62, and A's and C's together make 118; subtract 73 from 118, and add it to 62, and B will have 135, and A and C will have 45; and $45\times 3=135$: lastly, C's stock is 71, and B's and A's make 109; subtract 73 from 109, and add it to 71, and C will have 144, and A and B 36; and $36\times 4=144$.

Equ.

Equ. 1st, $a-2b-2c=-3e$.
2d, $3a-b+3c=\ 4e$.
3d, $4a+4b-e=\ 5e$.
4th, * $5b+9c=13e$.
5th, * $12b+7c=17e$.
6th, * * $73c=71e$.

Equ. 7th, * * $c=\frac{71e}{73}$.
8th, * b * $=\frac{62e}{73}$.
9th, a * * $=\frac{47e}{73}$.

PROBLEM 67.

96. *It is required to find four numbers,* a, b, c *and* d *so related to each other, that* a *with half the sum of all the rest may make* f; *that* b *with a third part of all the rest may make* f; *that* c *with a fourth part of all the rest may make* f; *and that* d *with a fifth part of all the rest may make* f.

SOLUTION.

The fundamental equations of this problem are

$$a+\frac{b+c+d}{2}=f. \qquad c+\frac{a+b+d}{4}=f.$$

$$b+\frac{a+c+d}{3}=f. \qquad d+\frac{a+b+c}{5}=f.$$

From which equations duly prepared, are derived the following;

Equ. 1, $2a+b+c+d=2f$.
2, $a+3b+c+d=3f$.
3, $a+b+4c+d=4f$.
4, $a+b+c+5d=5f$.
Twice the 2d — the first gives 5, * $5b+c+d=4f$.
The 2d — the 3d gives 6, * $2b-3c$ * $=-f$.
The 3d — the 4th gives 7, * * $3c-4d=-f$.
Twice the 5th — 5 times the 6th gives 8, * * $17c+2d=13f$.
Thrice the 8th — 17 times the 7th gives 9, * * * $74d=56f$.
This last gives 10, * * * $d=\frac{28f}{37}$.
The 7th and 10th give 11, * * c * $=\frac{25f}{37}$.
The 6th and 11th give 12, * b * * $=\frac{19f}{37}$.
The 1st, 12th, 11th and 10th give 13, a * * * $=\frac{1f}{37}$.

Make

Make $f=37$, and the numbers a, b, c, d, will come out 1, 19, 25, 28 reſpectively.

PROBLEM 68.

97. *Four gameſters,* A, B, C *and* D, *each with a different ſtock of money about him, but as yet unknown, ſit down to play; during which engagement,* A *wins half of* B'*s firſt ſtock,* B *wins a third part of* C'*s,* C *a fourth part of* D'*s, and* D *a fifth part of* A'*s; after this they all riſe with the ſame ſum of money about them, to wit,* g; *which* g, *though a known quantity, is not ſuppoſed to be determined till the operation is over: It is required to determine the ſeveral ſtocks of money with which the gameſters* A, B, C, D, *began to play; I mean with reſpect to the known quantity* g.

Various ſolutions may be given of this problem; but that which ſeems moſt proper to be inſerted here, is the following

SOLUTION.

Let a, b, c, d, repreſent the reſpective unknown ſums with which A, B, C and D began to play; then if A had only loſt a fifth part of his firſt ſtock to D, and had won nothing of B, his laſt ſtock would have been $\frac{a}{1}-\frac{a}{5}$, or $\frac{4a}{5}$; but according to the problem, A did not only loſe a fifth part of his ſtock to D, but alſo won half of B's firſt ſtock; therefore A's laſt ſum when the play was ended was $\frac{4a}{5}+\frac{b}{2}$; but the problem informs us, that when the play was ended, A's laſt ſum was g; whence we have this equation, $\frac{4a}{5}+\frac{b}{2}=g$; therefore $4a+\frac{5b}{2}=5g$; therefore $8a+5b=10g$: in like manner the firſt and ſecond conditions of the problem furniſh this equation, $\frac{b}{1}-\frac{b}{2}+\frac{c}{3}=g$, or $\frac{b}{2}+\frac{c}{3}=g$; which freed from fractions like the former, becomes $3b+2c=6g$. the ſecond and third conditions of the problem furniſh this equation, $\frac{c}{1}-\frac{c}{3}+\frac{d}{4}=g$, or $\frac{2c}{3}+\frac{d}{4}=g$; which freed from fractions like the reſt, becomes $8c+3d=12g$: laſtly, the laſt and firſt conditions of the problem alſo conſidered together, give this equation, $\frac{3d}{4}+\frac{a}{5}=g$; which in integral terms, ſtands thus, $15d+4a$, or $4a+15d=20g$: ſo that the

the fundamental equations of this problem when freed from fractions will ſtand thus;

$$8a+5b=10g.$$
$$3b+2c=6g.$$
$$8c+3d=12g.$$
$$4a+15d=20g.$$

Amongſt theſe equations it is plain there are but two of the firſt rank, to wit, the firſt, $8a+5b=10g$, and the laſt, $4a+15d=20g$; the ſecond fundamental equation in order, to wit, $3b+2c=6g$ is an equation of the ſecond rank; and the third in order, $8c+3d=12g$ is an equation of the third rank; therefore in the reſolution of this problem, theſe fundamental equations muſt not be written all together one under another, as in the former examples, but muſt be reſerved for their proper ranks, thus;

Equ. 1ſt, $8a+5b$ * * $=10g$.
2d, $4a$ * * $+15d=20g$.

Now as there are no other equations but theſe two of the firſt rank, proceed on to equations of the ſecond rank thus; according to art. 70, ſubtract twice the ſecond equation from the firſt, and there will remain $5b-30d=-30g$; whence dividing by 5, you will have $b-6d=-6g$ for a third equation, which ought to be placed regularly under the ſecond, thus;

Equ. 3d, * b * $-6d=-6g$.

But as this laſt equation is an equation of the ſecond rank, it will now be proper to draw out the equation of the ſecond rank hitherto kept in reſerve in the problem, to wit, $3b+2c=6g$, and to place it under the reſt for a fourth equation, thus;

Equ. 4th, * $3b+2c$ * $=6g$.

Subtract now three times the third equation from the fourth, that is, ſubtract * $3b$ * $-18d=-18g$ from * $3b+2c$ * $=6g$, and there remains $2c+18d=24g$; whence dividing by 2, you will have

Equ. 5th, * * $c+9d=12g$.

This is an equation of the third rank, and therefore it will now be proper to draw out the laſt equation of the fundamental ones given in the problem, which is of the ſame rank with this, and to place it under this for a ſixth equation, thus;

Equ. 6th, * * $8c+3d=12g$.

Subtract this ſixth equation from eight times the fifth, that is, * * $8c+3d=12g$ from * * $8c+72d=96g$, and there remains $69d=84g$; whence dividing by 3, we have

Equ. 7th, * * * $23d=28g$; and

Equ. 8th, * * * $d=\frac{28g}{23}$.

Sub-

Subſtitute this value for d in the fifth equation, and inſtead of $c+9d=12g$, you will now have $c+\frac{252g}{23}=12g$; whence $23c+252g=276g$; and $23c=24g$; and

$$\text{Equ. 9th, } * \quad * \quad c \quad * = \frac{24g}{23}.$$

Subſtitute the value of d found in the 8th equation inſtead of d in the 3d, where we had $b-6d=-6g$, and you will have $b-\frac{168g}{23}=-6g$; whence $23b-168g=-138g$; and $23b=30g$; and

$$\text{Equ. 10th, } * \quad b \quad * \quad * = \frac{30g}{23}.$$

Subſtitute this value for b in the firſt equation, $8a+5b=10g$, and you will have $8a+\frac{150g}{23}=10g$; whence $184a+150g=230g$; therefore $184a=80g$, and $a=\frac{80g}{184}=\frac{10g}{23}$; whence we have

$$\text{Equ. 11th, } a \quad * \quad * \quad * = \frac{10g}{23}.$$

So that at laſt all the quantities a, b, c, d, come to be known with reſpect to the quantity g; for $a=\frac{10g}{23}$, $b=\frac{30g}{23}$, $c=\frac{24g}{23}$, and $d=\frac{28g}{23}$: make $g=23$, and the quantities will all come out in whole numbers thus; $a=10$, $b=30$, $c=24$, and $d=28$; and they will anſwer the conditions of the queſtion; for at this rate

A's laſt ſum will be $10-2+15=23$;
B's laſt ſum will be $30-15+8=23$;
C's laſt ſum will be $24-8+7=23$;
and *D*'s laſt ſum will be $28-7+2=23$.

Equ. 1ſt,	$8a+5b$	$*$	$*=10g.$	Equ. 8th,	$* \; * \; * \quad d=\frac{28g}{23}.$
2d,	$4a$	$*$	$*+15d=20g.$	9th,	$* \; * \; c \quad * =\frac{24g}{23}.$
3d,	$* \quad b$	$*$	$-6d=-6g.$	10th,	$* \; b \; * \quad * =\frac{30g}{23}.$
4th,	$*$	$3b+2c$	$*=6g.$	11th,	$a \; * \; * \quad * =\frac{10g}{23}.$
5th,	$*$	$*$	$c+9d=12g.$		
6th,	$*$	$*$	$8c+3d=12g.$		
7th,	$*$	$*$	$* \quad 23d=28g.$		

The ſolution here given is deſigned to ſhew the learner how to proceed in caſes where all his fundamental equations are not of the ſame rank; for otherwiſe, this problem is capable of a much more elegant ſolution by the help of one letter only, as we ſhall find, if to avoid fractions, we put x for the money D won of A, thus;

D won of A,	x;
A's firſt ſtock,	$5x$;
Left A after his loſs to D,	$4x$:
A won of B,	$g-4x$;
B's firſt ſtock,	$2g-8x$;
Left B after his loſs to A,	$g-4x$:
B won of C,	$4x$;
C's firſt ſtock,	$12x$;
Left C after his loſs to B,	$8x$:
C won of D,	$g-8x$;
D's firſt ſtock,	$4g-32x$;
Left D after his loſs to C,	$3g-24x$.

D won of A, $24x-2g$; for the remainder of D's firſt ſtock after his loſs to C was $3g-24x$, and therefore if D had won nothing of A, this quantity $3g-24x$ would have been D's laſt ſum; but by the problem it appears that D's laſt ſum was g; therefore the money D won of A muſt be the exceſs of g above $3g-24x$; ſubtract therefore $3g-24x$ from g, and the remainder $24x-2g$ will be the money D won of A; and after the ſame manner were all the other winnings determined: here then we have two expreſſions for the money D won of A, to wit, x according to the ſuppoſition, and $24x-2g$ from the nature of the queſtion; therefore $24x-2g=x$; and $x=\frac{2g}{23}$: this being determined, the firſt ſtocks of all the gameſters will be determined by the poſitions, thus:

A's firſt ſtock,	$5x=\frac{10g}{23}$.	C's	$12x=\frac{24g}{23}$.
B's	$2g-8x=\frac{30g}{23}$.	D's	$4g-32x=\frac{28g}{23}$.

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ELEMENTS of ALGEBRA

BOOK III.

Some obſervations tending to the inveſtigation of the rule for extracting the ſquare root.

98. HAVING promiſed ſomewhere that I would take the firſt opportunity to account for the common method of extracting the ſquare root, I cannot think of any place more proper than this for the performance of my promiſe, not only as the practice of this rule grows upon our hands in the reſolution of quadratic equations, and therefore ought not, if poſſible, to be left any longer undemonſtrated, but more eſpecially conſidering that by this time our young Analyſt may be reaſonably ſuppoſed to be pretty well acquainted with ſome of the firſt rudiments of his art, which may enable him to follow me with more eaſe to himſelf, and perhaps to me too, than could poſſibly have been done at our firſt ſetting out. But here I muſt not omit to advertiſe my reader, that if a whole number, or a mixt number conſiſting of a whole number and decimal parts, be not a ſquare, it's ſquare root will conſiſt of an integral part, and of an infinite ſeries of decimal fractions: but if a rule can be found out for determining the integral part, it will be ſufficient, becauſe all the parts of the ſquare root of any number may be conſidered as integral till the operation is over, ſince it is not till then, that the diſtinction needs to be made between integral and decimal parts; ſee introduct. art. 24.

For the better effecting what I here propoſe, I ſhall lay down the following obſervations, which the learner muſt attend to, if he expects to go through the following demonſtration; and if there be ſtill any difficulties he may meet with or thinks he meets with in the application of theſe obſervations, the beſt advice I can give him here, as well as in many other parts of this book, is to read the demonſtration over and over again, by which means all the ſteps will become more familiar to him, and he will be the better able to put them together, in order to digeſt the whole.

OBSER-

OBSERVATION 1.

If any number, as $a+x$, consisting of two parts a and x, be squared, the product will be $aa+2ax+xx=aa+\overline{2a+x}\times x$.

OBSERVATION 2.

$1\times1=1$; therefore $10\times10=100$, and $100\times100=10000$, and $1000\times1000=1000000$, &c: whence I infer, that if any number consists of one or two places, it's square root, or at least the integral part of it, will consist but of one place; if a number consists of three or four places, the integral part of it's square root will consist of two places; if a number consists of five or six places, the integral part of it's square root will consist of three places, &c.: thus the integral part of the square root of this number of five places, to wit, 56644 will consist of three places; for the number 56644, lies between 10000, whose square root is 100, and 1000000, whose square root is 1000; therefore the integral part of the square root of the number 56644 must lie between 100 and 1000, and consequently must consist of three places.

OBSERVATION 3.

From the last observation it follows, that if of any number proposed, a point be put over the place of units, and another over the next place but one to the left hand, I mean that of hundreds, and another again over the next place but one to that, to wit, of ten thousands, and so on alternately; the number of points will discover the number of places of which the integral part of the square root of the proposed number consists: thus if the number 56644 be so pointed, it will stand thus, $\dot{5}6\dot{6}4\dot{4}$; and the three points shew, that the square root of this number, or the integral part of it at least, consists of three places.

OBSERVATION 4.

Supposing all things as in the last observation, if any number be advanced one point, that is, two places higher to the left hand, by adding cyphers, or by taking in decimals from the right hand, the figures expressing it's square root will still be the same, but will be advanced one place higher; if the number be advanced two points, the square root will be advanced two places; if three points, three places, &c: thus if the square root of the number 576 be 24, that of the number 57600 will be 240, that of 5760000 will be 2400, &c: thus again, if the square

ſquare root of the number 5.6644 be 2.38, the ſquare root of the number 566.44 will be 23.8, and that of the number 56644 will be 238, &c: whoever would be thoroughly convinced of this, as well as of the reaſon of the thing, may eaſily ſatisfy himſelf by ſquaring the numbers 2.38, 23.8, 238.

Note, that by ſquaring a number is here as well as every where elſe meant, multiplying it into itſelf, and not extracting it's ſquare root, as young beginners through inadvertency are very often apt to miſtake it.

OBSERVATION 5.

From the laſt obſervation it follows, that if the ſquare root of any number, ſuppoſe of 5.6644 lies between 2 and 3, the ſquare root of 566.44 will lie between 20 and 30; and that of 56644, between 200 and 300, &c: thus again, if the ſquare root of 566.44 lies between 23 and 24, the ſquare root of 56644 will lie between 230 and 240: this is plain; for ſuppoſe the ſquare root of 566.44 to be 23.8, then by the laſt obſervation, the ſquare root of 56644 will be 238; and as 23.8 lies between 23 and 24, ſo 238 lies between 230 and 240.

The inveſtigation of the rule for extracting the ſquare root.

99. Theſe things being obſerved, let now ſome number be propoſed, ſuch as is the number 56644; and let us ſee whether we cannot inveſtigate the ſquare root of this number by pure dint of reaſon, without any regard to, or helps from the common method. This number then being pointed according to the third obſervation, I ſhall firſt begin with the number 5, belonging to the firſt point to the left hand, ſetting aſide all the other figures, thus: the number 5 is itſelf no ſquare number, therefore I ſubtract the number 4 which is the neareſt ſquare number leſs than 5, from 5, and there remains 1; and as 2 is the ſquare root of 4, I conclude, that of all the infinite ſeries repreſenting the ſquare root of the number 5, the firſt term, or the integral part will be 2: again, as the ſquare of 2 is the neareſt ſquare number leſs than 5, which is no ſquare, it follows, that the ſquare of 3 will be the neareſt ſquare number greater than 5, and conſequently that the ſquare of 3, whatever it is, cannot be leſs than 6; whence I conclude in the next place, that the number 2 will be the firſt term of the ſquare root, not only of the number 5, but alſo of any other number between 5 and 6 at leaſt; therefore 2 will be the firſt figure expreſſing the ſquare root of the number 5.66, as alſo of the number 5.6644; therefore by the fourth obſervation, 2 will be the firſt

figure

figure of the ſquare root of the number 56644; which is one conſiderable point gained: now to diſcover the next figure in the root, I conſider the number 566 belonging to the two firſt dotts to the left hand, ſetting aſide all the other figures, thus; it has already been ſhewn that the firſt figure expreſſing the ſquare root of the number 5.66 is 2; therefore the ſquare root of the number 5.66 lies between 2 and 3; therefore by the fifth obſervation, the ſquare root of the number 566 lies between 20 and 30; let then $20+x$ repreſent the integral part of the ſquare root of the number 566, the letter x ſtanding for ſome whole number in the place of units; then it is plain, that the ſquare of $20+x$ muſt either be preciſely equal to 566 if 566 be an exact ſquare, or elſe it muſt be the neareſt ſquare number leſs than 566; but the ſquare of $20+x$ by the firſt obſervation is $20\times20+\overline{40+x}\times x$; therefore $20\times20+\overline{40+x}\times x$ is either equal to, or leſs than 566; ſubtract the ſquare of 20 from both ſides, by obſerving, that as before, 2×2 ſubtracted from 5 left 1, ſo now 20×20 ſubtracted from 500 will leave 100, and the ſame ſubtracted from 566, will leave 166; therefore $\overline{40+x}\times x$ muſt either be equal to, or leſs than 166, according as the number 566 is, or is not a perfect ſquare; and therefore if $\overline{40+x}\times x$ be leſs than 166, the difference will be the exceſs of the number 566 above the neareſt number leſs that is an exact ſquare: here we ſee, by the bye, why the number 166 is commonly called the reſolvend, to wit, becauſe it muſt be reſolved into two factors, x and $40+x$, the product of whoſe multiplication muſt either be equal to 166, or elſe it muſt be the neareſt product of the kind leſs: but to proceed; as it has been ſhewn already that the product $\overline{40+x}\times x$ is not to exceed 166, it follows, that the product $40x$ muſt be leſs than 166, and conſequently that x muſt be leſs than the quotient of 166 divided by 40, or of 16.6 divided by 4; but the quotient of 16.6 divided by 4 lies between the two whole numbers 4 and 5; therefore x is leſs than 5, and 4 is the greateſt whole number that can be ſuppoſed equal to x; let us then ſuppoſe x equal to 4, and let us ſee what will be the conſequence; now if $x=4$, we ſhall have $40+x=44$, and $\overline{40+x}\times x=44\times4=176$; therefore the ſuppoſition of $x=4$ was wrong, becauſe $\overline{40+x}\times x$ ought not to exceed 166; let us then ſuppoſe in the next place that $x=3$; then we ſhall have $40+x=43$, and $\overline{40+x}\times x=43\times3=129$, which is leſs than 166, and therefore not inconſiſtent with the foregoing conditions; and ſince x muſt be taken equal to the greateſt whole number the foregoing conditions will admit of, it follows, that x muſt be equal to 3, and conſequently that $20+x$,

or

or the integral part of the ſquare root of the number 566 is 23; and ſince $\overline{40+x}\times x$ or 129, ſubtracted from the reſolvend 166 leaves 37, it follows, that 23×23 ſubtracted from the number 566 will alſo leave 37: again, as the ſquare of 23 is the neareſt ſquare number leſs than 566 which is no ſquare, the ſquare of 24 muſt be greater than 566, and conſequently cannot be leſs than 567; therefore the ſquare root of the number 566.44 muſt neceſſarily lie between 23 and 24; therefore by the fifth obſervation, the ſquare root of the number 56644 muſt neceſſarily lie between 230 and 240; which is another ſtep made in the approximation: let us now ſuppoſe $230+x$ to be the ſquare root, or the integral part of the ſquare root of the number 56644; then by the firſt obſervation we ſhall have $230\times 230+\overline{460+x}\times x$ either equal to, or leſs than 56644: ſubtract 230×230 from both ſides, by obſerving, that as before 23×23 ſubtracted from 566 left 37, ſo now 230×230 ſubtracted from 56600 will leave 3700; and the ſame ſubtracted from 56644 will leave 3744; therefore $\overline{460+x}\times x$ muſt not exceed 3744; therefore $460x$ muſt be leſs than 3744; therefore x muſt be leſs than the quotient of 3744 divided by 460; but the quotient of 3744 divided by 460, or (which is pretty much the ſame thing, eſpecially with reſpect to the integral part) the quotient of 374 divided by 46 lies between the two whole numbers 8 and 9; therefore 8 is the greateſt whole number that can be ſuppoſed equal to x: let us then ſuppoſe $x=8$, and we ſhall have $\overline{460+x}=468$, and $\overline{460+x}\times x=468\times 8=3744$, which is juſt conſiſtent with the abovementioned conditions, and the number 238 is the exact ſquare root of the number propoſed 56644. *Q. E. I.*

Let now any one extract the ſquare root of the ſame number 56644 according to the common practice, and he will eaſily ſee, that the rule he there uſes is nothing elſe but a general obſervation drawn from this or the like inveſtigation.

The foundation of the common rule for extracting the cube root.

100. The cube of $\overline{a+x}$ is $a^3+3a^2x+3ax^2+x^3$, where $3a^2x+3ax^2+x^3$ is the reſolvend; make $\overline{3a+x}=b$, and you will have $3ax+xx=bx$, and $3a^2+3ax+xx=3a^2+bx$; and $3a^2x+3ax^2+x^3$, or the reſolvend $=\overline{3a^2+bx}\times x$; therefore the cube of $a+x$ is $a^3+\overline{3a^2+bx}\times x$: this anſwers to the firſt obſervation in the laſt article but one, and is the foundation of the rule for extracting the cube root; and whoſoever applies the reaſoning of the two laſt articles to the preſent

 caſe,

case, he will either be able of himself to form a rule for extracting the cube root, or at least he will be the better able to understand and remember the common rule as it is delivered in books of Arithmetic.

N. B. All the difficulty in extracting either the square or cube root, is in obtaining the two first figures of the root, wherein tryals are sometimes required; the rest of the process is more certain.

Of the composition and resolution of a square raised from a binomial root.

101. Hitherto we have been chiefly concerned in simple equations: it is now high time to apply ourselves to the resolution of quadratics; in order to which, something must be said concerning the nature of a binomial, upon which that resolution entirely depends.

Now a binomial (at least as it is here used) is a quantity consisting of two parts or members, connected together by the sign + or —, as $x+a$, $x-a$, $x+\frac{b}{2}$, $x-\frac{b}{2}$; and a square raised from a binomial root is nothing else but the square of such a quantity: thus the square of $x+\frac{b}{2}$ is $xx+bx+\frac{bb}{4}$, and that of $x-\frac{b}{2}$ is $xx-bx+\frac{bb}{4}$.

$$\begin{array}{l} x+\frac{b}{2} \\ x+\frac{b}{2} \\ \hline x^2+\frac{bx}{2}+\frac{bb}{4} \\ \quad +\frac{bx}{2} \\ \hline x^2+\frac{2bx}{2}+\frac{bb}{4} \end{array} \qquad \begin{array}{l} x-\frac{b}{2} \\ x-\frac{b}{2} \\ \hline x^2-\frac{bx}{2}+\frac{bb}{4} \\ \quad -\frac{bx}{2} \\ \hline x^2-\frac{2bx}{2}+\frac{bb}{4} \end{array}$$

$x^2+\frac{2bx}{2}+\frac{bb}{4}$; that is, $x^2+bx+\frac{bb}{4}$. $x^2-\frac{2bx}{2}+\frac{bb}{4}$; that is, $x^2-bx+\frac{bb}{4}$.

The difference betwixt these two squares arises from the different sign of b; and that only affects the second member; for the third member $\frac{bb}{4}$ will be the same, whether the quantity b be affirmative or negative; therefore if those cases be thrown into one, it will stand thus: *The square of*

of $x \pm \frac{b}{2}$ *is* $xx \pm bx + \frac{bb}{4}$; *to wit*, $+bx$ *when the root is* $x + \frac{b}{2}$, *and* $-bx$ *when the root is* $x - \frac{b}{2}$. Now of the three members that compose this square, the first xx is the square of x, the second $\pm bx$ is the root of that square multiplied into the coefficient $\pm b$; for the root of xx is x, and $x \times \pm b = \pm bx$; the third and last member $\frac{bb}{4}$, is the square of $\pm \frac{b}{2}$, that is, the square of half the coefficient of the second member; whence may be deduced the two following observations.

OBSERVATION 1.

Whenever we meet with a quantity consisting of two members, as $xx \pm bx$, *whereof one, as* xx, *is a square, and the other* $\pm bx$ *is the root of that square multiplied into some given coefficient* $\pm b$; *whenever I say we meet with such a quantity, it may be considered as an imperfect square raised from a binomial root, and may easily be compleated by adding* $\frac{bb}{4}$, *that is, by adding the square of half the coefficient of* x *in the second term:* thus $xx + 6x$ when compleated, becomes $xx + 6x + 9$; $xx - 8x$ when compleated becomes $xx - 8x + 16$; $xx + 3x$ when compleated becomes $xx + 3x + \frac{9}{4}$; for here the coefficient being 3, its half will be $\frac{3}{2}$, and the square of this will be $\frac{9}{4}$: again, $xx + \frac{2x}{3}$ when compleated becomes $xx + \frac{2x}{3} + \frac{1}{9}$; for here the second term is $\frac{2x}{3}$, and therefore the coefficient of x is $\frac{2}{3}$ by art. 70; but the half of $\frac{2}{3}$ is $\frac{1}{3}$, and the square of this is $\frac{1}{9}$: again, $xx - \frac{5x}{6}$ when compleated becomes $xx - \frac{5x}{6} + \frac{25}{144}$; for here the coefficient is $-\frac{5}{6}$, whose half is $-\frac{5}{12}$, and the square of this is $+\frac{25}{144}$: lastly, $xx - \frac{bx}{a}$ when compleated becomes $xx - \frac{bx}{a} + \frac{bb}{4aa}$; for here the coefficient is $-\frac{b}{a}$, it's half $-\frac{b}{2a}$, and the square of this is $\frac{bb}{4aa}$.

OBSERVATION 2.

In the second place it may be observed, that the root of such a square when compleated, that is, the root of $xx \pm bx + \frac{bb}{4}$ *will always be* $x \pm \frac{b}{2}$, *that is,*

is, it will always be the square root of the first member, together with half the coefficient of the second: thus the square root of $xx+6x+9$ will be $x+3$; that of $xx-8x+16$ will be $x-4$; that of $xx+3x+\frac{9}{4}$ will be $x+\frac{3}{2}$; that of $xx+\frac{2x}{3}+\frac{1}{9}$ will be $x+\frac{1}{3}$; that of $xx-\frac{5x}{6}+\frac{25}{144}$ will be $x-\frac{5}{12}$; and lastly, that of $xx-\frac{bx}{a}+\frac{bb}{4aa}$ will be $x-\frac{b}{2a}$.

The common form to which all quadratic equations ought to be reduced in order to be resolved.

102. Since an affected quadratic equation, as we have elsewhere defined it (art. 23,) is an equation consisting of three different sorts of quantities; one sort wherein the square of the unknown quantity is concerned, another sort wherein the unknown quantity is simply concerned, and a third sort wherein it is not concerned at all; it follows, that all quadratic equations whatever may be reduced to this form, *viz.* $Axx=Bx+C$; wherein A, B and C denote known integral quantities whether affirmative or negative, and x the quantity unknown, the sign $+$ on the latter side of the equation $Bx+C$, signifying no more than that the two quantities Bx and C are to be added together according to the common rules of addition, whether they be both affirmative, or both negative, or one affirmative and the other negative: this will easily be allowed, if it be considered, that quadratic equations, like all others, may be freed from fractions after the same manner as simple equations; and when that is done, there needs no more at most, than a bare transposition of the terms to reduce them to the form above described: we shall however give some examples of the reduction of quadratic equations to this form amongst those that follow.

A general theorem for resolving all quadratic equations.

103. This preparation being made, let now some general quadratic equation be proposed to be resolved, with which all particular equations may afterwards be compared, and by means whereof those equations may be more readily resolved; as for example, let the general equation in the last article be proposed, to wit, $Axx=Bx+C$; and let it be proposed to find the value or values of x in this equation: here transposing Bx, I have $Axx-Bx=C$; and then dividing by A in order to free

free xx the higheſt power of x from it's coefficient, I have $xx - \frac{Bx}{A} = \frac{C}{A}$; this done, I conſider the firſt ſide $xx - \frac{Bx}{A}$ as an imperfect ſquare raiſed from a binomial root; and accordingly I compleat that ſquare by art. 101, to wit, by adding $\frac{BB}{4AA}$, that is, by adding the ſquare of half the coefficient of the ſecond term; but if $\frac{BB}{4AA}$ muſt be added to the firſt ſide of the equation to compleat the ſquare, it muſt alſo be added to the other ſide to preſerve the equality, otherwiſe by an unequal addition, the equation would be deſtroyed: this equal addition then being made, the equation will ſtand thus, $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB}{4AA} + \frac{C}{A}$; but the two fractions $\frac{BB}{4AA}$ and $\frac{C}{A}$ when thrown into one, give $\frac{ABB + 4AAC}{4AAA}$, which dividing by A, gives $\frac{BB + 4AC}{4AA}$; therefore $xx - \frac{Bx}{A} + \frac{4AA}{BB} = \frac{BB + 4AC}{4AA}$; therefore the ſquare root of one ſide will be equal to the ſquare root of the other; but the ſquare root of the fraction $\frac{BB + 4AC}{4AA}$, at leaſt as it here ſtands in letters, cannot be extracted, becauſe, though the denominator $4AA$ be a ſquare, yet there is no literal quantity whatever which being multiplied into itſelf will produce $BB + 4AC$; therefore to put this numerator into the form of a ſquare, let us ſuppoſe $BB + 4AC = ss$; and then the equation will ſtand thus, $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{ss}{4AA}$; but the ſquare root of $xx - \frac{Bx}{A} + \frac{BB}{4AA}$ is $x - \frac{B}{2A}$, by art. 101; and the ſquare root of $\frac{ss}{4AA}$ is $\pm \frac{s}{2A}$, for a reaſon formerly given, to wit, becauſe $\frac{-s}{2A}$ when multiplied into itſelf will produce $\frac{+ss}{4AA}$ as well as $\frac{+s}{2A}$, and therefore by the very definition of the ſquare root, the former quantity has as good a right to be ſtiled the ſquare root of $\frac{ss}{4AA}$ as the latter; therefore this equation will

now

now be reduced to a ſimple one, and will ſtand thus, $x-\frac{B}{2A}=\pm\frac{s}{2A}$;
therefore $x=\frac{B\pm s}{2A}$, that is, $x=\frac{B+s}{2A}$, and $x=\frac{B-s}{2A}$. *Q. E. I.*

Thus we ſee that every quadratic equation neceſſarily admits of two numbers or roots (as they are called) which will equally anſwer the condition of the equation, that is, either of which being put equal to x, will make the two ſides of the equation equal one to the other; and theſe two roots in all arts and ſciences where quadratic equations are concerned, are of equal eſtimation, whether they be affirmative or negative, or one be affirmative and the other negative: as for example, in Geometry, if a line drawn from any point towards the right hand be conſidered as affirmative, a line drawn from the ſame point to the left hand ought to be conſidered as negative; for let AB be any line drawn from the fixt point A to the point B on the right hand, and then imagine the point B to move towards A; here then it is plain that the nearer B approaches towards A, the leſs will be the affirmative line AB; when the point B coincides with A, the line AB muſt be looked upon as nothing, and therefore when the point B by a continuation of it's motion has paſſed through A, ſo as to lye on the left hand of A, the line AB ought now to be looked upon as negative, having paſſed from ſomething through nothing into negation; and yet a line of this negative kind is as true a line as any of the affirmative kind; and therefore the negative roots of quadratic equations which exhibit negative lines, ought to be of equal eſtimation with the affirmative roots that exhibit affirmative lines; and the ſame will be the caſe (I ſay) of all other arts and ſciences where quadratic equations are concerned: but in common life, where negative quantities have no place, the affirmative roots of quadratic equations are only allowed of in the reſolution of problems, the negative ones being for the moſt part excluded.

N. B. 1*ſt*, The root of any quantity whether in numbers or letters, that cannot be expreſſed, is called a ſurd: thus $\sqrt{3}$ is a ſurd, and ſo alſo is $\sqrt{BB+4AC}$; and it was for this reaſon that I made $\sqrt{BB+4AC}=s$, or which is all one, $BB+4AC=ss$.

2*dly*, The quantity C and conſequently $4AC$ will ſometimes be negative; in which caſe the quantity ss, or $BB+4AC$ muſt be looked upon as the ſum of the affirmative quantity BB and the negative one $4AC$ when added together according to the common rules of addition.

3*dly*, In many of the following examples, the learner muſt be very careful to form a right eſtimation of negative quantities: thus for inſtance, if x, that is, $+x=-3$, he muſt make $4x$, or $+4x-3=-12$;
but

but he must make $-4x$, or $-4\times-3=+12$; so likewise $-x$, or $-1x$, or -1×-3 must be made equal to $+3$, &c.

A synthetical demonstration of the foregoing theorem.

104. In the last article it was demonstrated analytically, that if Axx be equal to $Bx+C$, then x must necessarily be equal both to $\frac{B+s}{2A}$, and to $\frac{B-s}{2A}$, supposing ss to be equal to $BB+4AC$. Now it may not be improbable but that the learner, especially if he has any taste or genius, may have a curiosity to see the same demonstrated again synthetically, that is, to see it demonstrated, that if x be made equal to $\frac{B+s}{2A}$, or $\frac{B-s}{2A}$, then Axx must necessarily be equal to $Bx+C$: it is therefore to gratify the learner in this particular, that I have added the following demonstration.

CASE 1st.

Let $x=\frac{B+s}{2A}$; then you will have $xx=\frac{BB+2Bs+ss}{4AA}$; multiply both sides by A, and you will have Axx (or one side of the general equation) equal to $\frac{BB+2Bs+ss}{4A}$; for a fraction may be multiplied by dividing the denominator, as well as by multiplying the numerator: again, since $x=\frac{B+s}{2A}$, you will have $Bx=\frac{BB+Bs}{2A}$; double both the numerator and denominator of this last fraction, which will not affect the value of the fraction, and you will have $Bx=\frac{2BB+2Bs}{4A}$; therefore $Bx+C=\frac{2BB+2Bs}{4A}+\frac{C}{1}=\frac{2BB+2Bs+4AC}{4A}=\frac{BB+2Bs+BB+4AC}{4A}=\frac{BB+2Bs+ss}{4A}$, because $BB+4AC=ss$ by the supposition; therefore $Axx=Bx+C$, since each side is equal to the same quantity $\frac{BB+2Bs+ss}{4A}$.

CASE

CASE 2d.

Let now $x=\frac{B-s}{2A}$, and you will have $xx=\frac{BB-2Bs+ss}{4AA}$, and Axx (or the first side of the general equation) $=\frac{BB-2Bs+ss}{4A}$: again, $Bx=\frac{BB-Bs}{2A}=\frac{2BB-2Bs}{4A}$; therefore $Bx+C=\frac{2BB-2Bs+4AC}{4A}=\frac{BB-2Bs+ss}{4A}$; therefore $Axx=Bx+C$, because each side is equal to the same quantity $\frac{BB-2Bs+ss}{4A}$.

Various examples of the resolution of affected quadratic equations, both with, and without the general theorem.

EXAMPLE I.

105. Let the equation proposed to be resolved be $6xx=5x-1$. This particular equation, as well as all those that follow, may be resolved after the same manner as the general one in art. 103: but as these resolutions are very often attended with fractions very troublesome to the young Analyst, and as these particular equations are nothing else but particular cases of the general one, it follows, that the resolution of these equations must necessarily be included in the resolution of the general one; and consequently, that these equations will be much more easily and readily resolved by referring them to the general one: however for the satisfaction of the learner, I shall resolve some of these equations both with, and without the general theorem; and first I shall resolve the equation proposed by the help of the general theorem thus; in the general equation, art. 103, we have $Axx=Bx+C$; in the particular one already proposed, we have $6xx=5x-1$; therefore A in the general equation answers to 6 in the particular one, B answers to 5, and C to -1; therefore if the particular equation be referred to the general one, it's resolution will be as follows: $A=6$, $B=5$, $C=-1$, $BB=25$, $4AC=-24$; therefore ss, or $BB+4AC$ will be the sum of 25 and $-24=1$; therefore $s=1$, $\frac{B+s}{2A}=\frac{5+1}{12}=\frac{1}{2}$, $\frac{B-s}{2A}=\frac{5-1}{12}=\frac{1}{3}$; therefore the two roots of this equation $6xx=5x-1$ are $\frac{1}{2}$ and $\frac{1}{3}$. The resolution of this equation in numbers without the general theorem, is as follows:

Equation,

Equation, $6xx = 5x - 1$; therefore $6xx - 5x = -1$, and $xx - \frac{5x}{6} = -\frac{1}{6}$; where $xx - \frac{5x}{6}$ may be confidered as the two firft members of a fquare raifed from a binomial root; the coefficient of the fecond term is $\frac{-5}{6}$, it's half $\frac{-5}{12}$, and the fquare of this $\frac{+25}{12 \times 12}$, which expreffion I choofe to make ufe of rather than $\frac{+25}{144}$ for a reafon that will prefently be feen; add now $\frac{25}{12 \times 12}$ to both fides, that is, to one fide to compleat the fquare, and to the other to preferve the equality, and you will have $xx - \frac{5x}{6} + \frac{25}{12 \times 12} = \frac{-1}{6} + \frac{25}{12 \times 12}$; here now it is certain that the fractions $\frac{-1}{6}$ and $\frac{+25}{12 \times 12}$ muft be reduced to the fame denomination in order to be added together into one fum; but if this be done the common way, it will be impoffible to obtain the fquare root of that fum without a further reduction; therefore to avoid this, I enquire what number the denominator 6 muft be multiplied by to make it 12×12 the fame with the other denominator, and the anfwer in this cafe, as well as in all others of this kind will be very eafy; for $2 \times 6 = 12$, and therefore $12 \times 2 \times 6$, or $24 \times 6 = 12 \times 12$; therefore I multiply both the numerator and denominator of the fraction $\frac{-1}{6}$ into 24, and fo have $\frac{-24}{12 \times 12}$; and this added to the other fraction $\frac{+25}{12 \times 12}$ gives $\frac{+1}{12 \times 12}$; and now the equation will be $xx - \frac{5x}{6} + \frac{25}{12 \times 12} = \frac{1}{12 \times 12}$; extract the root of both fides, and you will have $x - \frac{5}{12} = \pm \frac{1}{12}$, whence $x = \frac{5 \pm 1}{12}$; but $\frac{5+1}{12} = \frac{1}{2}$, and $\frac{5-1}{12} = \frac{1}{3}$; therefore $x = \frac{1}{2}$, or $\frac{1}{3}$.

This may alfo be proved fynthetically thus: let $x = \frac{1}{2}$, then you will have $xx = \frac{1}{4}$, and $6xx = \frac{6}{4}$ or $1\frac{1}{2}$: again, $5x = \frac{5}{2} = 2\frac{1}{2}$; therefore $5x - 1 = 1\frac{1}{2}$; therefore $6xx = 5x - 1$, fince each equals $1\frac{1}{2}$.

Let us now fuppofe $x = \frac{1}{3}$, and you will have $xx = \frac{1}{9}$, and $6xx = \frac{6}{9}$, or $\frac{2}{3}$: on the other hand you will have $5x = \frac{5}{3}$ or $1\frac{2}{3}$; therefore $5x - 1 = \frac{2}{3}$; therefore $6xx = 5x - 1$: thefe two fractions therefore will anfwer the con-

condition of the equation; and there are no other numbers beſide theſe, whether whole numbers or fractions, that will do it.

EXAMPLE 2.

Let the equation to be reſolved be $24x - 2xx = xx + 45$. Here tranſpoſing $-2xx$ we have $3xx + 45 = 24x$, whence $3xx = 24x - 45$; and thus we have reduced the equation propoſed to the form of the general one in art. 102; wherefore applying that general equation to this particular one, the reſolution, by art. 103 will be as follows: $A = 3$, $B = 24$, $C = -45$, $BB = 576$, $4AC = -540$, $ss = 576 - 540 = 36$, $s = 6$, $\frac{B+s}{2A} = 5$, $\frac{B-s}{2A} = 3$; therefore $x = 5$, or 3; and this will further eaſily appear by ſubſtituting 5 or 3 for x in the original equation thus; $x = 5$; therefore $24x = 120$; $xx = 25$; therefore $24x - 2xx = 120 - 50 = 70$, which is one ſide of the equation: on the other ſide we have $xx + 45 = 25 + 45 = 70$; therefore $24x - 2xx = xx + 45$. Again, let $x = 3$, then we ſhall have $24x = 72$, and $xx = 9$, and $24x - 2xx = 54$: on the other hand, $xx + 45 = 54$; therefore $24x - 2xx = xx + 45$.

N. B. This laſt equation when reduced to the form of the general one in art. 102, ſtood thus; $3xx = 24x - 45$: but this equation might have been reduced to a more ſimple one of the ſame form by dividing the whole by 3, and then the equation would have ſtood thus, $xx = 8x - 15$; in which caſe we ſhould have had $A = 1$, $B = 8$, $C = -15$, $BB = 64$, $4AC = -60$, $ss = 4$, $s = 2$, $\frac{B+s}{2A} = 5$, $\frac{B-s}{2A} = 3$, as before: the ſolution of the foregoing equation in the common way is this; $xx - 8x = -15$; therefore compleating the ſquare, $xx - 8x + 16 = 1$; therefore extracting the ſquare root, $x - 4 = \pm 1$ therefore $x = +4 \pm 1 = 5$, or 3.

EXAMPLE 3.

Let the equation to be reſolved be $72x - 2xx + 144 = 3xx - 8x + 444$. Here by tranſpoſitions we have $72x + 144 = 5xx - 8x + 444$, and $80x + 144 = 5xx + 444$, and $5xx = 80x - 300$, and $xx = 16x - 60$; which equation being reſolved like that in the laſt example, gives $x = 10$, or 6; which may alſo be eaſily ſeen by ſubſtituting 10 or 6 for x in the original equation.

EXAMPLE 4.

Let the equation to be reſolved be $28x - xx = 115$. Here we have $xx + 115 = 28x$, and $xx = 28x - 115$; which equation being reſolved

ſolved like that in the ſecond example, gives $x=23$, or 5; the proof whereof is eaſy.

EXAMPLE 5.

Let the equation to be reſolved be $\frac{120}{x}-5=\frac{120}{x+4}$; therefore $120-5x=\frac{120x}{x+4}$; therefore $100x-5xx+480=120x$; therefore $5xx+120x=100x+480$; therefore $5xx=-20x+480$; therefore (dividing by 5) $xx=-4x+96$; therefore in this caſe, $A=1$, $B=-4$, $C=96$, $BB=16$, $4AC=384$, $ss=16+384=400$, $s=20$, $\frac{B+s}{2A}=\frac{-4+20}{2}=8$, $\frac{B-s}{2A}=\frac{-4-20}{2}=-12$; therefore in this equation, $x=8$, or -12: the proof is thus; let $x=+8$; then $\frac{120}{x}=15$, and $\frac{120}{x}-5=10$: again, $x+4=12$, and $\frac{120}{x+4}=10$; therefore $\frac{120}{x}-5=\frac{120}{x+4}$. Again, let $x=-12$, then $\frac{120}{x}=-10$; therefore, $\frac{120}{x}-5=-10-5=-15$: on the other hand, $x+4=-12+4=-8$; therefore $\frac{120}{x+4}=\frac{120}{-8}=-15$; therefore $\frac{120}{x}-5=\frac{120}{x+4}$. The reſolution in the common way is this; $xx=-4x+96$; therefore $xx+4x=96$; therefore $xx+4x+4=100$; therefore $x+2=\pm 10$; therefore $x=-2\pm 10=+8$, or -12.

EXAMPLE 6.

Let the equation to be reſolved be $2xx+3x=65$; therefore $2xx=-3x+65$; therefore in this caſe, $A=2$, $B=-3$, $C=65$, $BB=9$, $4AC=520$, $ss=529$, $s=23$, $\frac{B+s}{2A}=\frac{-3+23}{4}=5$, $\frac{B-s}{2A}=\frac{-3-23}{4}=-6\frac{1}{2}$: therefore in this equation, $x=+5$, or $-6\frac{1}{2}$: that $x=+5$ will eaſily be ſeen; and that $x=-6\frac{1}{2}$, or that $-6\frac{1}{2}$ being ſubſtituted for x will make $2xx+3x=65$, I thus demonſtrate: $x=-6\frac{1}{2}=\frac{-13}{2}$; therefore $xx=\frac{+169}{4}$; therefore $2xx=$

 $\frac{169}{2}$

$\frac{169}{2}$; and $+3x=+3\times\frac{-13}{2}=\frac{-39}{2}$; therefore $2xx+3x=\frac{169-39}{2}=\frac{130}{2}=65$. The resolution in numbers; $2xx+3x=65$; therefore $xx+\frac{3x}{2}+\frac{9}{4\times4}=\frac{65}{2}+\frac{9}{4\times4}=\frac{520+9}{4\times4}=\frac{529}{4\times4}$; therefore $x+\frac{3}{4}=\pm\frac{23}{4}$; therefore $x=\frac{-3\pm23}{4}=+5$, or $-6\frac{1}{2}$.

EXAMPLE 7.

Let the equation to be resolved be $9xx-x=140$; therefore $9xx=1x+140$. Here $A=9$, $B=1$, $C=140$, $BB=1$, $4AC=5040$, $ss=5041$, $s=71$, $\frac{B+s}{2A}=4$, $\frac{B-s}{2A}=-3\frac{8}{9}$; therefore $x=+4$, or $-3\frac{8}{9}$: the latter case I thus demonstrate; $x=-3\frac{8}{9}=\frac{-35}{9}$; therefore $xx=\frac{+1225}{81}$; therefore $9xx=\frac{1225}{9}$: again, $-1x$, that is, $-1\times\frac{-35}{9}=\frac{+35}{9}$, therefore $9xx-x=\frac{1225+35}{9}=\frac{1260}{9}=140$. In numbers thus; $9xx-1x=140$; therefore $xx-\frac{1x}{9}=\frac{140}{9}$; therefore $xx-\frac{1x}{9}+\frac{1}{18\times18}=\frac{140}{9}+\frac{1}{18\times18}=\frac{5040+1}{18\times18}=\frac{5041}{18\times18}$; extract the root of both sides, that is, of $xx-\frac{1x}{9}+\frac{1}{18\times18}$ on one side, and of $\frac{5041}{18\times18}$ on the other, and you will have $x-\frac{1}{18}=\pm\frac{71}{18}$; whence $x=+4$, or $-3\frac{8}{9}$.

EXAMPLE 8.

Let the equation to be resolved be $\frac{45}{2x+3}+\frac{116}{4x+5}=7$; therefore $45+\frac{232x+348}{4x+5}=14x+21$; therefore $180x+225+232x+348=56xx+154x+105$; that is, $412x+573=56xx+154x+105$; therefore $258x+573=56xx+105$; therefore $56xx=258x+468$; therefore (dividing by 2) you have $28xx=129x+234$; which equation being compared with the general one exhibited in art. 103, gives

$A=28$,

$A=28$, $B=129$, $C=234$, $BB=16641$, $4AC=26208$, $ss=42849$, $s=207$, $\frac{B+s}{2A}=6$, $\frac{B-s}{2A}=-1\frac{11}{28}$; therefore in this equation $x=+6$, or $-1\frac{11}{28}$; both which I thus demonſtrate: firſt $x=6$; therefore $2x+3=15$; therefore $\frac{45}{2x+3}=3$; moreover, $4x+5=29$; therefore $\frac{116}{4x+5}=4$; therefore $\frac{45}{2x+3}+\frac{116}{4x+5}=3+4=7$: ſecondly, $x=-1\frac{11}{28}=-\frac{39}{28}$; therefore $2x=\frac{-39}{14}$; therefore $2x+3=\frac{-39}{14}+\frac{3}{1}=\frac{3}{14}$; therefore $\frac{45}{2x+3}$ is the quotient of $\frac{45}{1}$ divided by $\frac{3}{14}$; but this quotient, according to the rules of fractional diviſion, is $\frac{630}{3}=210$; therefore $\frac{45}{2x+3}=210$: again, $4x=\frac{-39}{7}$; therefore $4x+5=\frac{-39}{7}+\frac{5}{1}=\frac{-4}{7}$; therefore $\frac{116}{4x+5}$ is the quotient of $\frac{116}{1}$ divided by $\frac{-4}{7}$; but this quotient is $\frac{-812}{4}$, or -203; therefore $\frac{116}{4x+5}=-203$; therefore $\frac{45}{2x+3}+\frac{116}{4x+5}=210-203=7$.

The reſolution of this equation in the common way is as follows; $56xx-258x=468$; therefore $xx-\frac{258x}{56}=\frac{468}{56}$: here the coefficient of the ſecond term is $-\frac{258}{56}$, it's half $-\frac{129}{56}$, and the ſquare of this $\frac{16641}{56\times56}$; add this ſquare to both ſides, and you will have $xx-\frac{258x}{56}+\frac{16641}{56\times56}=\frac{468}{56}+\frac{16641}{56\times56}=\frac{26208+16641}{56\times56}=\frac{42849}{56\times56}$; extract the ſquare root of both ſides, that is, of $xx-\frac{258x}{56}+\frac{16641}{56\times56}$ on one ſide, and of $\frac{42849}{56\times56}$ on the other, and you will have $x-\frac{129}{56}=\pm\frac{207}{56}$; whence $x=+6$, or $-1\frac{11}{28}$.

EXAM-

EXAMPLE 9.

Let the equation be $15x - xx = 56$; then this equation being resolved by the general theorem gives $x = 8$, or 7; and in the common way it is thus resolved; $15x - xx = 56$; change all the signs to make xx affirmative, and you will have $xx - 15x = -56$; whence $xx - 15x + \frac{225}{4} = -56 + \frac{225}{4} = \frac{1}{4}$; therefore $x - \frac{15}{2} = \pm\frac{1}{2}$, and $x = 8$, or 7: but what I chiefly intend by this example is to shew, that in resolving a quadratic equation by the general theorem there is no necessity of making any transposition to exhibit xx affirmative when it would otherwise have been negative; as for instance, in the equation here proposed we had $15x - xx = 56$; transpose $15x$, and you will have $-xx$, that is, $-1xx = -15x + 56$; let this equation be referred to the general one in art. 102, and resolved by the general theorem in art. 103, and you will have $A = -1$, $B = -15$, $C = 56$, $BB = 225$, $4AC = -224$, $ss = 1$, $s = 1$, $\frac{B+s}{2A} = \frac{-15+1}{-2} = \frac{-14}{-2} = +7$, $\frac{B-s}{2A} = \frac{-15-1}{-2} = +8$.

How the learner is to proceed when the roots of a quadratic equation are inexpressible.

106. As there are but few square numbers in comparison of the rest, and as all quadratic equations are resolved by extracting the square root, it follows, that there are but few quadratic equations capable of an exact numeral solution in comparison of those that are not: but as the square root may be extracted to any degree of exactness we please, the resolution of a quadratic equation, which depends upon it, may also be performed to any degree of accuracy whatever; as will appear by the following example.

EXAMPLE 10.

Let the equation be $xx - 4x + 1 = 0$, or $xx = 4x - 1$. Here $A = 1$, $B = 4$, $C = -1$, $BB = 16$, $4AC = -4$, $ss = 12$, $s = \sqrt{12}$, $\frac{B+s}{2A} = \frac{4+\sqrt{12}}{2}$, and $\frac{B-s}{2A} = \frac{4-\sqrt{12}}{2}$; therefore $x = \frac{4+\sqrt{12}}{2}$, or $\frac{4-\sqrt{12}}{2}$: but let us enquire in the next place, whether these two

fractions

fractions are not capable of being reduced to more ſimple terms; firſt then it is plain that $\frac{4}{2}=2$, and I ſay further that $\frac{\sqrt{12}}{2}=\sqrt{3}$; for 12 $=3\times4$; therefore $\sqrt{12}=\sqrt{3}\times\sqrt{4}=\sqrt{3}\times2$; therefore $\frac{\sqrt{12}}{2}=\sqrt{3}$; whence it follows, that $x=2+\sqrt{3}$, or $2-\sqrt{3}$; but $\sqrt{3}$ extracted to three decimal places gives 1 .732: therefore $2+\sqrt{3}=3.732$, and $2-\sqrt{3}=.268$; therefore $x=$ (nearly) 3 .732, or .268, as will be further evident from the proof following: firſt $x=3.732$; therefore $xx=13.927824$; and $4x=14.928$; therefore $4x-xx=1.000176$; therefore $xx-4x=-1.000176$; therefore $xx-4x+1=-.000176$ $=0$ very nearly: ſecondly, let $x=.268$ and you will have $xx=.071824$ and $4x=1.072$, and $4x-xx=1.000176$; therefore $xx-4x=-1.000176$; therefore $xx-4x+1=-.000176=0$ very nearly; therefore in both caſes, the condition of the equation is anſwered to as many figures or cyphers, as is equal to the number of decimal places to which the ſquare root of 3 was extracted.

It may ſeem to ſome perhaps a paradox to aſſert, that though the two ſurd values of the unknown quantity found in this and the like caſes, are not to be expreſſed in numbers, yet they may be demonſtrated to be juſt: Thus I ſhall demonſtrate, that if either of the two values of x found in the laſt caſe, to wit, $2+\sqrt{3}$, or $2-\sqrt{3}$, be ſubſtituted for x, we ſhall have this equation $xx-4x+1=0$, which was the equation there propoſed: in order to this, make $\sqrt{3}=s$; and firſt, let $x=2+\sqrt{3}$, or $2+s$; and we ſhall have $xx=4+4s+ss$, and $-4x=-8-4s$; and $xx-4x=4+4s+ss-8-4s=ss-4$; but if $s=\sqrt{3}$, $ss=3$, and $ss-4=-1$; therefore, $xx-4x=-1$, and $xx-4x+1=0$: ſecondly, let $x=2-\sqrt{3}$, or $2-s$, and we ſhall have $xx=4-4s+ss$, and $-4x=-8+4s$, and $xx-4x=ss-4=-1$, as before; whence $xx-4x+1=0$.

Of impoſſible roots in a quadratic equation, and whence they ariſe.

107. The roots of quadratic equations are not only very often inexpreſſible, but ſometimes even impoſſible, as will appear by the following example.

EXAMPLE II.

Let the equation be $xx-4x+6=0$, or $xx=4x-6$. Here $A=1$, $B=4$, $C=-6$, $BB=16$, $4AC=-24$, $ss=-8$, $s=\sqrt{-8}$, $\frac{B+s}{2A}=\frac{4+\sqrt{-8}}{2}$, $\frac{B-s}{2A}=\frac{4-\sqrt{-8}}{2}$; but $\frac{4}{2}=2$, and

and $-8=-2\times+4$; therefore $\sqrt{-8}=\sqrt{-2}\times\sqrt{+4}=\sqrt{-2}$ $\times 2$; therefore $\frac{\sqrt{-8}}{2}=\sqrt{-2}$; therefore in this equation, $x=2+\sqrt{-2}$, or $2-\sqrt{-2}$; but as no quantity whatever, either affirmative or negative, being multiplied into itſelf will produce a negative, it follows, that $\sqrt{-2}$ is not only an inexpreſſible quantity, but alſo an impoſſible one; and conſequently, that the two values of x in this equation $2+\sqrt{-2}$ and $2-\sqrt{-2}$ will both be impoſſible.

N. B. Though the roots of this laſt equation be impoſſible in their own natures, yet they may be abſtractedly demonſtrated to be juſt, as in the laſt article, by making $s=\sqrt{-2}$, and conſequently $ss=-2$.

From what has been ſaid concerning impoſſible roots, it appears that one root of a quadratic equation can never be impoſſible alone, but that they muſt either be both poſſible or both impoſſible: for it appears from the reſolution of the laſt equation, that the impoſſibility of the roots flows from the impoſſibility of the quantity s, or of the ſquare root of ss when it is negative; now when s is poſſible, both the roots of the equation $\frac{B+s}{2A}$ and $\frac{B-s}{2A}$ will be poſſible; on the other hand, when s is impoſſible, both the roots muſt neceſſarily be impoſſible.

Since the poſſibility or impoſſibility of the two roots of a quadratic equation depends upon the quantity ss being affirmative or negative, it follows, that when ss and conſequently s equals nothing; the roots will be in the limit between poſſible and impoſſible: now if $s=0$, we ſhall have $\frac{B+s}{2A}=\frac{B}{2A}$, and $\frac{B-s}{2A}=\frac{B}{2A}$; therefore the two unequal roots of a quadratic equation grow nearer and nearer to a ſtate of equality as they grow nearer and nearer to a ſtate of impoſſibility, but do not come to be equal till they come to the limit between poſſibility and impoſſibility.

How to find the ſum and product of the two roots of a quadratic equation without reſolving it: alſo how to generate a quadratic equation that ſhall have any two given numbers whatever for it's roots.

108. *In a quadratic equation of this general form, to wit,* $Axx=Bx+C$, *the ſum of the roots will always be* $\frac{B}{A}$, *and the product of their multi-*

multiplication $\frac{-C}{A}$: for the roots of ſuch an equation were $\frac{B+s}{2A}$ and $\frac{B-s}{2A}$; the ſum whereof is $\frac{2B}{2A}$, or $\frac{B}{A}$; and if theſe two roots be multiplied together, their product will amount to $\frac{BB-ss}{4AA}$; but $ss = BB+4AC$ as was formerly ſuppoſed, art. 103; therefore $ss-BB = 4AC$, and $BB-ss = -4AC$; therefore $\frac{BB-ss}{4AA}$, or the product of the two roots equals $\frac{-4AC}{4AA} = \frac{-C}{A}$.

Therefore if $A=1$, that is, if the equation be $xx = Bx+C$, the ſum of the roots will be B, and their product $-C$; that is, as the equation now ſtands, the ſum of the roots will be the coefficient of the unknown quantity on the ſecond ſide of the equation, and their product, what we call the abſolute term, with it's ſign changed.

Hence we have an eaſy way to form a quadratic equation whoſe roots ſhall be any two given numbers whatever: as for inſtance, ſuppoſe I would have a quadratic equation whoſe roots ſhall be the two numbers 3 and 4; here it is plain that the ſum of the two numbers 3 and 4 is 7, and that the product of their multiplication is 12; therefore I form an equation whereof one ſide is xx, and the other ſide is $7x-12$, to wit, $xx = 7x-12$; and the roots of this equation will be the given numbers 3 and 4, as will appear from the reſolution: if I intend the two roots to be 3 and -4, their ſum will be -1, and the product of their multiplication -12, and the equation $xx = -x+12$: if the roots are to be -3 and $+4$, their ſum will be $+1$, the product of their multiplication -12, and the equation $xx = x+12$: laſtly, if the roots are to be -3 and -4, their ſum will be -7, the product of their multiplication $+12$, and the equation $xx = -7x-12$. I ſhall demonſtrate one general caſe according to the reſolution given in art. 103, which will be ſufficient to ſhew the way to all the reſt: let then the roots propoſed be p and q, whoſe ſum is $p+q$, and the product of whoſe multiplication is pq; and the equation will be $xx = \overline{p+q} \times x - pq$; now if this equation be referred to the general one, we ſhall have $A=1$, $B = \overline{p+q}$, $C = -pq$, $BB = pp+2pq+qq$, $4AC = -4pq$, $ss = pp-2pq+qq$, $s = p-q$, $\frac{B+s}{2A} = \frac{p+q+p-q}{2} = \frac{2p}{2} = p$, $\frac{B-s}{2A}$

$=\frac{p+q-p+q}{2}=\frac{2q}{2}=q$; therefore the two roots of this equation are p and q. *Q. E. D.*

I think I ought not to omit here, that if any one has a mind to form a quadratic equation with any two given impoſſible roots whatever, (if I may be allowed the expreſſion,) it may be done by the foregoing rule, provided that theſe impoſſible roots be in ſuch a form as is proper for a quadratic equation: as for example, ſuppoſe I would form a quadratic equation with theſe two impoſſible roots, to wit, $2+\sqrt{-3}$ and $2-\sqrt{-3}$, I put ss for -3; for though no poſſible quantity multiplied into itſelf can produce a negative, yet an impoſſible one may, that being the very thing wherein the impoſſibility conſiſts; making then $ss=-3$, I have $s=\sqrt{-3}$, and ſo the two roots of the equation will now be $\overline{2+s}$, and $\overline{2-s}$; the ſum of theſe two roots is 4, and the product of their multiplication $4-ss$; but if $ss=-3$, $-ss=+3$, and $4-ss=4+3=7$; therefore the equation with theſe roots will be $xx=4x-7$: and this will be further evident by the reſolution; for if $xx=4x-7$, that is, if $xx-4x=-7$, we ſhall have $xx-4x+4=-3$, and $x-2=\pm\sqrt{-3}$, and $x=2+\sqrt{-3}$, or $2-\sqrt{-3}$.

How to determine the ſigns of the poſſible roots of a quadratic equation without reſolving it.

109. If all the terms of a quadratic equation be thrown on one ſide of the equation, ſo as to be made equal to nothing; and if the term wherein xx, the ſquare of the unknown quantity is concerned, be made the firſt, that wherein x, the ſimple power is concerned, be made the ſecond, and the abſolute term, as it is called, be made the third; the number of affirmative and negative roots in ſuch an equation may be found by the following rule, to wit, *As often as the ſigns are changed in paſſing through all the terms from the firſt to the laſt, of ſo many affirmative roots will the equation conſiſt; but as often as the ſigns are the ſame, ſo many negative roots will be found in the equation.* This is true in all equations whatever, though at preſent we ſhall only demonſtrate it in the caſe of a quadratic equation: but firſt we ſhall give the following explication of the rule.

CASE I.

Let the equation be $axx-bx+c=0$. Here there are two changes in paſſing through the terms from the firſt to the laſt, to wit, from

$+axx$

$+axx$ to $-bx$, and from $-bx$ to $+c$; therefore the roots of this equation are both affirmative.

CASE 2.

Let the equation be $axx-bx-c=0$. Here from $+axx$ to $-bx$ is one change, and from $-bx$ to $-c$ is none; therefore this equation consists of an affirmative and a negative root.

CASE 3.

Let the equation be $axx+bx-c=0$. Here in passing from $+axx$ to $+bx$, there is no change of sign, but in passing from $+bx$ to $-c$ there is a change; therefore this equation also consists of an affirmative and a negative root.

CASE 4.

Lastly, let the equation be $axx+bx+c=0$. Here there are no changes, and consequently the roots of this equation are both negative. All these cases I shall demonstrate in the following manner.

CASE 1.

Let the equation be $axx-bx+c=0$, or $axx=bx-c$. Here the product of the two roots is $\frac{c}{a}$ by the last article, that is, the product of the two roots is an affirmative quantity, and therefore those roots must either be both affirmative or both negative; but they cannot be both negative, because their sum is $\frac{+b}{a}$, by the same article; therefore they must both be affirmative.

CASE 2.

Let the equation be $axx-bx-c=0$, or $axx=bx+c$. Here the product of the two roots is $-\frac{c}{a}$ and consequently those roots must be of different kinds, one affirmative and the other negative; and because their sum, $+\frac{b}{a}$, is an affirmative quantity, it is an argument that the greater root is affirmative.

CASE 3.

Let the equation be $axx+bx-c=0$, or $axx=-bx+c$. Here again the product of the two roots is $\frac{-c}{a}$, which argues one root to be

 affir-

affirmative and the other negative; and because their sum $\frac{-b}{a}$ is a negative quantity, it is an indication that of these two roots, the greater is the negative one.

CASE 4.

Lastly, let the equation be $axx+bx+c=0$, or $axx=-bx-c$. Here the product of the two roots is $+\frac{c}{a}$ an affirmative quantity; therefore the roots are either both affirmative or both negative; but they cannot be both affirmative, because their sum $\frac{-b}{a}$ is negative; therefore they must both be negative.

Impossible roots excluded out of the foregoing rule.

The rule here given for determining the number of affirmative and negative roots relates only to possible roots; for impossible ones cannot be said to belong to any class, either of affirmatives or negatives; nay so capricious are they in this respect, that in one and the same equation, the very same impossible roots shall sometimes appear under one form, and sometimes under the other: as for example, this equation $xx+3=0$ may be filled up two ways without affecting either the equation or it's roots; to wit, either thus, $xx-0x+3=0$, the roots of which equation according to the foregoing rule are both affirmative; or thus, $xx+0x+3=0$, the roots of which equation, though it be the same with the other, and differs only in form, are both negative: the reason of this absurdity is, that the two roots of the equation $xx+3=0$ are impossible, and occasioned this confusion by putting on one shape in one equation, and another shape in the other: this will further appear from the resolution; for if $xx+3=0$, we have $xx=-3$, and $x=+\sqrt{-3}$, or $-\sqrt{-3}$, which are both impossible quantities. Again, the equation $x^3-3=0$ may be filled up various ways; as thus, $x^3-0x^2+0x-3=0$, in which equation, according to the foregoing rule, there are three affirmative roots; or thus, $x^3-0x^2-0x-3=0$, in which equation there is but one affirmative root and two negative ones: hence an experienced Analyst would immediately conclude (as is really the case) that two of the roots of the equation $x^3-3=0$ were impossible, and that they stood for affirmative quantities in the former way of putting the equation, and for negative ones in the latter. This will further appear, when we come to treat of cubic equations.

Of

Of biquadratics, and other equations in the form of quadratics.

110. Thus much for the resolution, nature, and properties of a quadratic equation: I shall only add an example or two more of other equations that sometimes put on the form of quadratics, and have done.

EXAMPLE 12.

Let the equation to be resolved be, $\frac{1600}{xx}+xx=116$; therefore $1600+x^4=116xx$; therefore $x^4=116xx-1600$. This equation is, properly speaking, a biquadratic, that is, an equation wherein the fourth power of the unknown quantity is concerned: now as every possible quadratic equation has two roots, which will equally answer the condition thereof, so a cubic equation, that is, an equation that rises to the third power of the unknown quantity may have three such roots, a biquadratic four, &c: but the equation $x^4=116xx-1600$, though it be a biquadratic, and admits of four roots, yet it is in the form of a quadratic, if we consider xx as the unknown quantity; in which case x^4 must be looked upon as the square of the unknown quantity, and the equation must be referred to the general one in art. 103, thus; $A=1$, $B=116$, $C=-1600$, $BB=13456$, $4AC=-6400$, $ss=7056$, $s=84$, $\frac{B+s}{2A}=100$, $\frac{B-s}{2A}=16$; therefore in this equation, $xx=100$, or 16: now if $xx=100$, we shall have $x=+$ or -10; if $x^2=16$, we shall have $x=+$ or -4; therefore the four roots of this biquadratic equation are, $+10$, -10, $+4$ and -4: but though in this equation x has four significations, xx has but two, *viz.* 100 and 16, either of which being substituted instead of xx in the original equation, will answer that equality, as may easily be tryed.

N. B. Whenever of the four roots of a biquadratic equation any two are equal and contrary to the other two, the equation will be in form of a quadratic, and may be resolved accordingly.

EXAMPLE 13.

Let the equation be $\frac{576}{xx}-xx=55$: here we have $576-x^4=55xx$, and $x^4+55xx=576$, and $x^4=-55x^2+576$; therefore according to the general equation in art. 103, $A=1$, $B=-55$, $C=576$, $BB=3025$, $4AC=2304$, $ss=5329$, $s=73$, $\frac{B+s}{2A}=9$,

$\frac{B-s}{2A}$

$\frac{B-s}{2A} = -64$; therefore in this equation, $xx = +9$ or -64: if $xx = +9$, $x = +$ or -3; if $xx = -64$, x will be equal to $+\sqrt{-64}$, or $-\sqrt{-64}$, both which values are impoſſible; ſo that in this equation x has but two values, $+$ or -3, the other two being impoſſible; and xx has two values, to wit, $+9$ and -64, which are both poſſible, and which being ſubſtituted inſtead of xx into the original equation, will anſwer that equality. From this example it is eaſy to ſee, that a biquadratic equation may have four roots, and never can have more; yet it may ſometimes have fewer, upon the account of ſome of it's roots becoming impoſſible; nay inſtances might eaſily be given wherein all the roots of a biquadratic equation are impoſſible.

If any one diſapproves of the reſolutions here given, he may perhaps reliſh the following better: let the equation be $Ax^4 = Bx^2 + C$; here putting z for xx, and conſequently zz for x^4, the equation will be changed into this common quadratic, $Azz = Bz + C$; which being reſolved, z or xx, and conſequently x itſelf will be known: ſuppoſe the equation to be $Ax^6 = Bx^3 + C$; here putting z for x^3, the equation will be changed into a quadratic, as before, to wit, $Azz = Bz + C$, the reſolution whereof will give z or x^3, and conſequently x by an extraction of the cube root: laſtly, let the equation be $Ax = B \times \sqrt{x} + C$; here putting zz for x, and z for $\sqrt{x}$, the equation will be $Azz = Bz + C$, as before; whence z, and conſequently zz or x will be known.

The ſolution of ſome problems producing quadratic equations.

PROBLEM 69.

111. *It is required to divide the number* 60 *into two ſuch parts, that the product of their multiplication may amount to* 864.

SOLUTION.

Put x for one of the parts; then will the other part be $60 - x$, and the product of their multiplication will be $60x - xx$; whence the equation will be $60x - xx = 864$; therefore $xx + 864 = 60x$, and $xx = 60x - 864$: this equation compared with the general one in art. 103, gives $A = 1$, $B = 60$, $C = -864$, $BB = 3600$, $4AC = -3456$, $ss = 144$, $s = 12$, $\frac{B+s}{2A} = 36$, $\frac{B-s}{2A} = 24$; therefore the parts ſought are 24 and 36; which upon tryal will anſwer the conditions of the problem.

Obſerva-

Obſervations upon the foregoing problem.

OBSERVATION 1ſt.

In this problem we may clearly ſee the neceſſity of the unknown quantity's having ſometimes two diſtinct values in one and the ſame equation: for here, if I put x for the greater part of 60, the leſs will be $60-x$, and the equation will be $60x-xx=864$: ſuppoſe now I put x for the leſs part; then the greater will be $60-x$, and the equation will ſtill be $60x-xx=864$; therefore, whether x be put for the greater or the leſs part, we ſtill fall into the ſame equation $60x-xx=864$; whence I infer, that this equation muſt either give us both the parts ſought, or neither; ſince no reaſon can be ſhewn why it ſhould give us one part rather than the other.

OBSERVATION 2d.

Hence alſo we ſee the neceſſity ſometimes of impoſſible roots, to wit, when the caſes of problems to be ſolved by them become impoſſible: as for inſtance, if any number, as 60, be divided into two parts, the nearer the two parts approach towards an equality, the greater will be the product of their multiplication; and therefore if the parts be equal, the product will be the greateſt poſſible: thus if the parts be 24 and 36, the product will be 864; if they be 25 and 35, the product will be 875; if 30 and 30 the product will be 900, which will be the greateſt poſſible: let us now for once put an impoſſible caſe, and let it be required to divide the number 60 into two ſuch parts that the product of their multiplication may amount to 901; here the equation will be $60x-xx=901$; which being reſolved according to art. 103, gives $x=\frac{60+\sqrt{-4}}{2}$, or $\frac{60-\sqrt{-4}}{2}$; but theſe values of x may be reduced to more ſimple terms thus; $-4=-1\times+4$ therefore $\sqrt{-4}=\sqrt{-1}\times\sqrt{+4}=\sqrt{-1}\times2$; therefore $\frac{\sqrt{-4}}{2}=\sqrt{-1}$; but $\frac{60}{2}=30$; therefore the two parts ſought are $30+\sqrt{-1}$, and $30-\sqrt{-1}$, both which are impoſſible upon the account of the impoſſibility of $\sqrt{-1}$; and yet theſe two parts abſtractedly conſidered will anſwer the conditions of the problem; for if $\sqrt{-1}$ be made equal to s, the two parts will be $30+s$ and

and $30 - s$; whose sum is 60, and the product of whose multiplication is $900 - ss$; but if $s = \sqrt{-1}$, we shall have $ss = -1$, and $-ss = +1$, and $900 - ss = 901$; therefore the product of the two parts, $30 + \sqrt{-1}$, and $30 - \sqrt{-1}$ amount to 901, as was required.

OBSERVATION 3d.

Lastly, we here also see the necessity of both the roots of a quadratic equation becoming impossible at once. Two impossible quantities added together, may sometimes make a possible one, because one quantity may be as much impossible one way as the other is the contrary way: thus the two impossible quantities $30 + \sqrt{-1}$ and $30 - \sqrt{-1}$ being added together make 60, the impossible surds $+\sqrt{-1}$ and $-\sqrt{-1}$ destroying one another; but a possible and an impossible quantity when added together can never make a possible one; and therefore the two parts of 60 in this problem must either be both possible, or both impossible.

PROBLEM 70.

112. *There are three numbers in continual proportion, whereof the middle term is sixty, and the sum of the extremes one hundred twentyfive: What are the extremes?*

SOLUTION.

For the extremes put x and $125 - x$, and you will have this proportion; x is to 60 as 60 is to $125 - x$, whence by multiplying extremes and means, you have this equation, $125x - xx = 3600$, or $xx + 3600 = 125x$, or $x^2 = 125x - 3600$: here then $A = 1$, $B = 125$, $C = -3600$, $BB = 15625$, $4AC = -14400$, $ss = 1225$, $s = 35$, $\frac{B+s}{2A} = 80$, $\frac{B-s}{2A} = 45$; therefore in this equation, $x = 45$, or 80; but x represents either extreme, because, which extreme soever x is put for, the other will be $125 - x$, and the same equation will arise, to wit, $125x - xx = 3600$; therefore the two extremes are 45 and 80; and they will answer the conditions of the problem; for 45 is to 60 as $\frac{45}{15}$ is to $\frac{60}{15}$, that is, as 3 to 4; and 60 is to 80 as $\frac{60}{20}$ is to $\frac{80}{20}$, which is also as 3 to 4.

PROBLEM 71.

113. *It is required, having given the sum or the difference of two numbers, together with the sum of their squares, to find the numbers.*

SOLU-

SOLUTION.

Case 1st. Let the sum of the numbers sought be 28, and the sum of their squares 400; then putting x and $28-x$ for the two numbers sought, the square of the former will be xx, the square of the latter $784-56x+xx$, and the sum of their squares $2xx-56x+784=400$; and the same equation will arise, whether x be made to stand for one number or the other; therefore the two values of x in this equation will be the two numbers sought; but if $2xx-56x+784=400$, we shall have $2xx-56x=-384$; divide the whole by 2 for a more simple equation, and you will have $xx-28x=-192$; and $xx=28x-192$; which equation being resolved according to art. 103, gives $x=12$, or 16; therefore 12 and 16 are the two numbers sought.

Case 2d. Let now the difference of two numbers be given, suppose 4, and let the sum of their squares be 400, as before; then putting x for the less number, and $x+4$ for the greater, the sum of their squares will be $2xx+8x+16=400$; whence $2xx+8x=384$, $xx+4x=192$, $xx+4x+4=196$, $x+2=\pm 14$, $x=+12$ or -16; now it cannot be supposed that $+12$ and -16 are the two numbers required in the problem, for their difference is 30, not 4; neither ought it to be expected; for when x was put for the less number, and $x+4$ for the greater, the equation was $2xx+8x+16=400$; but if x be put for the greater number, and consequently $x-4$ for the less, the equation will be $2xx-8x+16=400$, different from the former; since then a different equation arises according as x is put for the greater or less number, it cannot be expected that one and the same equation should give both: the true state of the case is this; there are two pairs of numbers which will equally solve this question, and the equation $2xx+8x+16=400$ gives the lesser number of each pair; for if we make $x=12$, and $x+4=16$, the numbers 12 and 16 will solve the problem; on the other hand, if we make $x=-16$, we shall have $x+4=-12$, and the numbers -16 and -12 will equally solve the problem; for their difference is $+4$, and the sum of their squares $+400$: here then we may observe, that affirmative and negative solutions of problems are of equal estimation in the nature of things, though perhaps not amongst men, the narrowness of our minds contracting our views; but truth does justice alike to all: certainly negative numbers differ no more from affirmative ones, than affirmative ones do from one another, which is in degree, not in kind; and therefore in the nature of things, negative quantities ought no more to be excluded out of the scale of number, than affirmative ones, though in common life they are set aside.

PROBLEM 72.

114. *What two numbers are those, whose sum is seventeen, and the sum of their cubes one thousand three hundred forty three?*

SOLUTION.

For the two numbers sought put x and $17-x$, and the cube of the former will be xxx, and the cube of the latter $4913-867x+51xx-xxx$, as appears from the following computation:

$$\begin{array}{l} 17-x \\ 17-x \\ \hline 289-17x+xx \\ \quad -17x \\ \hline 289-34x+xx \\ 17-x \\ \hline 4913-578x+17xx-x^3 \\ \quad -289x+34xx \\ \hline 4913-867x+51xx-x^3. \end{array}$$

Therefore the sum of these two cubes will be $51xx-867x+4913=1343$, and the equation will be the same, whichsoever of the two numbers sought x is made to stand for; but if $51xx-867x+4913=1343$, we shall have $51xx-867x=-3570$; divide the whole by 51, which though not necessary, is however convenient, to render the equation more simple, since it may be done without fractions, and you will have, $xx-17x=-70$; which being reduced as in art. 103, gives $x=7$, or 10; therefore 7 and 10 are the two numbers sought.

PROBLEM 73.

115. *Let there be a square whose side is a hundred and ten inches; it is required to assign the length and breadth of a rectangled parallelogram or long square, whose perimeter shall be greater than that of the square by four inches, but whose area shall be less than the area of the square by four square inches.*

N. B. By the perimeter of a plain figure is meant the length of a line that will encompass it round; so that the perimeter of a square is equal to four times it's side; and the perimeier of a rectangled parallelogram is equal to twice it's length and twice it's breadth added together.

SOLUTION.

Since the side of the given square is 110 inches, it's area will be 12100 square inches; therefore the area of the parallelogram sought will

will be 12096 ſquare inches: again, the perimeter of the given ſquare is 440 inches; therefore the perimeter of the parallelogram ſought muſt be 444 inches; therefore half it's perimeter, or it's length and breadth added together muſt be 222 inches; therefore, if either the length or breadth be called x, the other will be $222 - x$, and the area will be $222x - xx = 12096$; which equation reſolved according to art. 103, will give $x = 96$, or 126; therefore the breadth of the parallelogram ſought muſt be 96 inches, and the length 126 inches: and theſe numbers will anſwer the conditions of the queſtion; for twice the length will be 252, twice the breadth 192, and the whole perimeter 444; moreover 126×96, or the area will be 12096, as the problem requires.

SCHOLIUM.

This problem ſhews how groſſly they are miſtaken who think to eſtimate the areas or magnitudes of plain figures by their perimeters, as if ſuch figures were greater or leſs in proportion as their perimeters were ſo; whereas here we ſee, that the perimeter of one figure may be greater than that of another by four inches, and at the ſame time it's area may be leſs than the area of that other by four ſquare inches. This error, it is true, does not obtain but in low and vulgar minds, nor there neither any longer than whileſt it continues to be a matter of mere ſpeculation, and truth and falſhood are equally indifferent to them: for whenever men come to apply their notions, and find it their intereſt not to be miſtaken, then it is, and frequently not till then, that they begin to look about them, correct their errors, and entertain more juſt and accurate notions of things. The greateſt part of mankind have a natural averſion to abſtract thinking, and where their intereſt is not concerned, will rather ſubmit their opinions to humour, caprice and cuſtom, or be content to be without any opinions at all, than they will examine ſtrictly into the nature of things.

PROBLEM 74.

116. *One buys a certain number of Oxen for eighty guineas; where it muſt be obſerved, that if he had bought four more for the ſame money, they would have come to him a guinea apiece cheaper: What was the number of oxen?*

SOLUTION.

For the number of oxen put x; then to find the price of a ſingle ox, ſay, if x oxen coſt 80 guineas, what will one ox coſt? and the anſwer is $\frac{80}{x}$; and for the ſame reaſon, if he had bought 4 more, that is, $x+4$

for the ſame money, the price of an ox would have been $\frac{80}{x+4}$; but according to the problem, the latter price is leſs than the former by one guinea; whence we have this equation, $\frac{80}{x} - 1 = \frac{80}{x+4}$; therefore $80 - x = \frac{80x}{x+4}$; therefore $\overline{80-x} \times \overline{4+x}$ or $320 + 76x - xx = 80x$ therefore $xx + 80x = 76x + 320$; therefore $xx = -4x + 320$. Here then $A = 1$, $B = -4$, $C = 320$, $BB = 16$, $4AC = 1280$, $ss = 1296$, $s = 36$, $\frac{B+s}{2A} = 16$, $\frac{B-s}{2A} = -20$; therefore $x = +16$, or -20; therefore the number of oxen was 16, the negative root -20 having no place in this problem; and this number 16 anſwers the condition of the problem; for if 16 oxen coſt 80 guineas, one will coſt 5 guineas; but if 20 oxen coſt 80 guineas, one will coſt 4 guineas.

N. B. The equation $\frac{80}{x} - 1 = \frac{80}{x+4}$, gave $x = +16$ or -20, not becauſe the number -20 would ſolve the problem, but becauſe it would ſolve the equation; for if we make $x = -20$, we ſhall have $\frac{80}{x} = -4$, and $\frac{80}{x} - 1 = -5$: on the other ſide we ſhall have $x + 4 = -16$, and $\frac{80}{x+4} = -5$; therefore if x be made equal to -20, we ſhall have $\frac{80}{x} - 1 = \frac{80}{x+4}$, becauſe both ſides are equal to -5; and ſo in all other caſes we ſhall always find, that the ſeveral roots of an equation will be ſuch as will equally ſolve that equation, though perhaps they may not be equally proper to ſolve the problem from whence the equation was deduced: but of this more in another place.

PROBLEM 75.

117. *A certain company at a tavern had a reckoning of ſeven pounds four ſhillings to pay; upon which two of the company ſneaking off, obliged the reſt to pay one ſhilling apiece more then they ſhould have done: What was the number of perſons?*

SOLUTION.

For the number of perſons put x; then to find the number of ſhillings every man ſhould have paid, ſay, if x perſons were to have paid 144 ſhillings,

ſhillings, what muſt one man have paid? and the anſwer is $\frac{144}{x}$; therefore $\frac{144}{x}$ is the number of ſhillings every man ſhould have paid; and for the ſame reaſon $\frac{144}{x-2}$ is the number of ſhillings every man did pay; but according to the problem, this latter reckoning is greater than the former by one ſhilling; whence the equation will be $\frac{144}{x}+1=\frac{144}{x-2}$; therefore $144+x=\frac{144x}{x-2}$; therefore $\overline{x-2}\times\overline{144+x}$, or $xx+142x-288=144x$; therefore $xx-288=2x$; therefore $xx=2x+288$. Here then $A=1$, $B=2$, $C=288$, $BB=4$, $4AC=1152$, $ss=1156$, $s=34$, $\frac{B+s}{2A}=18$, $\frac{B-s}{2A}=-16$; therefore $x=+18$, or -16; but negative roots have no place in this ſort of problems; therefore the number of perſons was 18, which anſwers the condition; for $\frac{144}{18}=8$, and $\frac{144}{16}=9$.

PROBLEM 76.

118. *What number is that, which being added to it's ſquare root will make two hundred and ten?*

SOLUTION.

For the number ſought put xx; then will it's ſquare root be x, and the equation will be $xx+x=210$, or $xx=-x+210$; where $A=1$, $B=-1$, $C=210$, $BB=1$, $4AC=840$, $ss=841$, $s=29$, $\frac{B+s}{2A}=14$, $\frac{B-s}{2A}=-15$; therefore $x=+14$, or -15; therefore xx or the number ſought, equals 196 or 225, ſuppoſing the ſquare root of 225 to be -15; and either of theſe two numbers will anſwer the condition; for $196+14=210$, and $225-15=210$.

PROBLEM 77.

119. *What two numbers are thoſe, the product of whoſe multiplication is one hundred ninety two, and the ſum of whoſe ſquares is ſix hundred and forty?*

SOLU-

SOLUTION.

For the two numbers sought put x and $\frac{192}{x}$; then will the square of the former be xx, and that of the latter $\frac{36864}{xx}$, and the sum of their squares will be $xx+\frac{36864}{xx}=640$; which equation will be the same, whichsoever of the two numbers sought x is made to stand for; but if $xx+\frac{36864}{xx}=640$, we shall have $x^4+36864=640xx$; and $x^4=640x^2-36864$: here then $A=1$, $B=640$, $C=-36864$, $BB=409600$, $4AC=-147456$, $ss=262144$, $s=512$, $\frac{B+s}{2A}=576$, $\frac{B-s}{2A}=64$; therefore $xx=576$, or 64; therefore $x=+$ or -24, or $+$ or -8; therefore the two numbers sought are 8 and 24.

PROBLEM 78.

120. *One lays out a certain sum of money in goods, which he sold again for twenty four pounds, and gained as much* per cent *as the goods cost him: I demand what they cost him.*

N. B. One's gain *per cent* is so much as he gains, every hundred pounds he lays out; or if he does not lay out so much as a hundred pounds, his gain *per cent* however, is so much as he would have gained if he had laid out a hundred pounds with the same advantage: thus if he lays out 20 pounds and gains 2 pounds, he is said to make 10 *per cent* of his money, because 20 pounds is to 2 pounds as 100 pounds is to 10 pounds.

SOLUTION.

Put x for the money laid out, and the gain will be $24-x$; say then by the golden rule, if in laying out x he gained $24-x$, what would he have gained if he had laid out 100 pounds to the same advantage? and the answer will be $\frac{2400-100x}{x}$; therefore $\frac{2400-100x}{x}$ will be his gain *per cent*; but according to the problem, this gain is equal to x, the money laid out; therefore $x=\frac{2400-100x}{x}$, and $xx=2400-100x$: here then $A=1$, $B=-100$, $C=2400$, $BB=10000$, $4AC=9600$,

$=9600$, $ss=19600$, $s=140$, $\frac{B+s}{2A}=20$, $\frac{B-s}{2A}=-120$; therefore the money laid out was 20 pounds; therefore his gain *per* 20 was 4 pounds; therefore his gain *per cent* was 20 pounds, equal to the money laid out.

PROBLEM 79.

121. *One lays out thirty three pounds fifteen ſhillings in cloth, which he ſold again for forty eight ſhillings* per *piece, and gained as much in the whole as a ſingle piece coſt: I demand how he bought in his cloth* per *piece.*

SOLUTION.

Put x for the number of ſhillings every ſingle piece was bought for, and the gain *per* piece will be $48-x$; ſay then by the rule of proportion, if in laying out x he gained $48-x$, what did he gain in laying out 33 pounds 15 ſhillings, or 675 ſhillings? and the anſwer will be $\frac{32400-675x}{x}$; therefore $\frac{32400-675x}{x}$ will be his whole gain; but according to the problem, the whole gain was equal to x, the money given for a ſingle piece; therefore $x=\frac{32400-675x}{x}$ therefore $xx=32400-675x$; therefore $A=1$, $B=-675$, $C=32400$, $BB=455625$, $4AC=129600$, $ss=585225$, $s=765$, $\frac{B+s}{2A}=45$, $\frac{B-s}{2A}=-720$; therefore $x=+45$, or -720; therefore the money every ſingle piece was bought for, was 45 ſhillings, and the gain *per* piece was 3 ſhillings; but if 45 ſhillings gains 3 ſhillings, 33 pounds 15 ſhillings, or 675 ſhillings, will gain 45 ſhillings; therefore the whole gain was 45 ſhillings, equal to the money given for a ſingle piece.

N. B. It is not impoſſible but that ſometimes two different problems may produce one and the ſame equation; and then the equation muſt provide equally for both: therefore in ſuch a caſe, though the equation has two roots, and both affirmative, yet it muſt not be expected that both roots ſhould equally ſerve for the ſolution of one problem, and that there ſhould be no ſolution left for the other; we ought rather to conclude, whenever an equation gives two roots, and both affirmative, whereof one only will ſolve the problem that produced the equation, we ought, I ſay, rather to conclude, that the other root is for the ſolution of ſome other problem producing the ſame equation; a curious inſtance whereof we have in the two following problems.

PRO-

PROBLEM 80.

122. *Two travellers* A *and* B, *ſet out from two places* C *and* D *at the ſame time,* A *from* C *bound for* D, *and* B *from* D *bound for* C; *when they met and had computed their travels, it was found, that* A *had travelled thirty miles more than* B, *and that at their rate of travelling,* A *expected to reach* D *in four days, and* B *to reach* C *in nine days: I demand the diſtance between the two places* C *and* D.

SOLUTION.

Put x for the number of miles between C and D; then it is plain that A and B both together had travelled x miles when they met; therefore as much as the miles travelled by A exceeded $\frac{x}{2}$, juſt ſo much did the miles travelled by B come ſhort of $\frac{x}{2}$; but by the ſuppoſition, A's miles exceeded thoſe of B by 30; therefore A muſt have travelled $\frac{x}{2}$ $+ 15$ or $\frac{x+30}{2}$ miles; and B muſt have travelled $\frac{x}{2} - 15$ or $\frac{x-30}{2}$ miles; therefore the remaining part of A's journey is $\frac{x-30}{2}$ miles, which he expects to perform in four days, and the remaining part of B's journey is $\frac{x+30}{2}$ miles, which he expects to perform in 9 days: theſe things being allowed, let us now enquire into the number of days each hath travelled already; and firſt for A ſay, if A expects to travel $\frac{x-30}{2}$ miles in 4 days, in how many days did he travel $\frac{x+30}{2}$ miles? and the anſwer is $\dfrac{4 \times \frac{x+30}{2}}{\frac{x-30}{2}} = \dfrac{4 \times \overline{x+30}}{x-30}$; then for B ſay, if B expects to travel $\frac{x+30}{2}$ miles in 9 days, in how many days did he travel $\frac{x-30}{2}$ miles? and the anſwer is $\dfrac{9 \times \overline{x-30}}{x+30}$; therefore A hath travelled $\dfrac{4 \times \overline{x+30}}{x-30}$ days,

days, and B $\frac{9\times\overline{x-30}}{x+30}$ days from the time of their firſt ſetting out; but as they both ſet out at the ſame time, and are now met, they muſt both have travelled the ſame number of days; therefore $\frac{4\times\overline{x+30}}{x-30}=\frac{9\times\overline{x-30}}{x+30}$ ∴ multiply both ſides of the equation into $\overline{x-30}$, and you will have $4\times\overline{x+30}=\frac{9\times\overline{x-30}\times\overline{x-30}}{x+30}$; again, multiply by $\overline{x+30}$, and you will have $4\times\overline{x+30}\times\overline{x+30}=9\times\overline{x-30}\times\overline{x-30}$; extract the ſquare root of both ſides, and you will have $\pm 2\times\overline{x+30}=\pm 3\times\overline{x-30}$: this general equation reſolves itſelf into 4 particular ones, *viz.*

1ſt, $+2\times\overline{x+30}=+3\times\overline{x-30}$.
2d, $+2\times\overline{x+30}=-3\times\overline{x-30}$.
3d, $-2\times\overline{x+30}=+3\times\overline{x-30}$.
4th, $-2\times\overline{x+30}=-3\times\overline{x-30}$.

But as the two laſt of theſe equations give but the ſame values as the two former, I ſhall only make uſe of the two former, thus;

1ſt, Suppoſe $+2\times\overline{x+30}=+3\times\overline{x-30}$, then we ſhall have $2x+60=3x-90$, and $x=150$.

2dly, Suppoſe $+2\times\overline{x+30}=-3\times\overline{x-30}$, then we ſhall have $2x+60=-3x+90$, and $x=6$; therefore the diſtance between the two places C and D muſt either be 150 miles, or 6 miles; but 6 miles it cannot be, becauſe when A came up to B, he had travelled 30 miles more than B, and had not yet reached D; therefore the diſtance between the two places C and D muſt be 150 miles; which will ſatisfy the problem; for then A muſt have travelled $75+15$, or 90 miles, and B $75-15$ or 60 miles, from the time of their ſetting out; therefore A has 60 miles, and B 90 to travel; but if A could travel 60 miles in 4 days, he muſt, at the ſame rate, have travelled 90 miles in 6 days, and if B could travel 90 miles in 9 days, he muſt have travelled 60 miles alſo in 6 days; therefore they both travelled the ſame number of days from the time of their firſt ſetting out to the time of their meeting, as the problem requires.

PROBLEM 81.

123. *Two travellers* A *and* B, *ſet out from two places* C *and* D *at the ſame time;* A *from* C *with a deſign to paſs through* D, *and* B *from* D *with a deſign to travel the ſame way: after* A *had overtaken* B, *and they had computed their travels, it was found, that they had both together travelled thirty miles, that* A *had paſſed through* D *four days before, and that* B *at his rate of travelling, was a nine days journey diſtant from* C: *I demand the diſtance between the two places* C *and* D.

SOLUTION.

Put x for the number of miles from C to D; then it is plain, that A muſt have travelled more miles than B by x; but they both together travelled 30 miles, by the ſuppoſition; therefore as much as A's miles exceeded 15, juſt ſo much muſt B's miles come ſhort of 15; but the whole difference was x, as above; therefore A muſt have travelled $15+\frac{x}{2}$, or $\frac{30+x}{2}$ miles, and B muſt have travelled $15-\frac{x}{2}$, or $\frac{30-x}{2}$ miles; therefore A's diſtance from D, after he had overtaken B, was $\frac{30-x}{2}$ miles, which he had travelled in 4 days, and B's diſtance from C was $\frac{30+x}{2}$ miles, which by the problem he could travel in 9 days; therefore to find how many days each had travelled already, ſay, if A hath travelled $\frac{30-x}{2}$ miles from D in 4 days, in how many days did he travel $\frac{30+x}{2}$ miles ſince his departure from C? and the anſwer is $\frac{4\times\frac{30+x}{2}}{\frac{30-x}{2}}=\frac{4\times\overline{30+x}}{30-x}$; again ſay, if B could travel $\frac{30+x}{2}$ miles, the whole diſtance from C, in 9 days, in how many days did he travel $\frac{30-x}{2}$ miles ſince his ſetting out from D? and the anſwer is $\frac{9\times\overline{30-x}}{30+x}$; but as they both ſet out at the ſame time, and A has now overtaken B, they muſt both have travelled the ſame number of days; therefore we have this

this equation, $\frac{4\times\overline{30+x}}{30-x}=\frac{9\times\overline{30-x}}{30+x}$: multiply both ſides into $\overline{30-x}$, and you will have $4\times\overline{30+x}=\frac{9\times\overline{30-x}\times\overline{30-x}}{30+x}$; again, multiply by $\overline{30+x}$, and you will have $4\times\overline{30+x}\times\overline{30+x}=9\times\overline{30-x}\times\overline{30-x}$; but the product of $\overline{30-x}\times\overline{30-x}$ differs nothing from the product of $\overline{x-30}\times\overline{x-30}$, as will appear upon tryal, and will be further evident from hence, that $\overline{30-x}$ and $\overline{x-30}$ differ no more from one another than an affirmative quantity does from an equal negative one, and therefore each multiplied into itſelf muſt give the ſame product; therefore the equation as it now ſtands is, $4\times\overline{x+30}\times\overline{x+30}=9\times\overline{x-30}\times\overline{x-30}$; but this equation is the ſame with the equation deduced from the laſt problem, which juſtifies what I obſerved before, art. 121, that different problems may produce the ſame equation; therefore the two roots of this equation will be 6 and 150, as in the laſt article; therefore the diſtance between the two places *C* and *D* muſt either be 6 miles, or 150 miles; but 150 miles it cannot be, becauſe, after *A* had paſſed from *C* beyond *D*, and at laſt had overtaken *B*, they had both travelled but 30 miles; therefore the diſtance from *C* to *D* muſt be 6 miles; and this number will anſwer the conditions of the problem; for then *A*, when he had overtaken *B*, had travelled $15+3$ or 18 miles, and *B* $15-3$ or 12 miles; therefore *A* had got 12 miles beyond *D* in 4 days time, and *B* was 18 miles diſtant from *C*, which he could travel in 9 days; but at the rate of 12 miles in 4 days, *A* muſt have performed his 18 miles journey in 6 days; and at the rate of 18 miles in 9 days, *B* muſt have performed his 12 miles journey alſo in 6 days; therefore from the time of their firſt ſetting out to the time of *A*'s overtaking *B*, they had both travelled the ſame number of days, as the problem requires; therefore the ſuppoſition whereupon this calculation was founded, to wit, that the diſtance of *C* from *D* was 6 miles, is juſt.

N. B. The ſolutions here given of the two laſt problems, are in my opinion, the moſt natural, though ſomewhat different from the reſt.

A LEMMA.

124. *The ſum of a ſeries of quantities in arithmetical progreſſion may be had, by adding the greateſt and leaſt terms together, and then multiplying either half that ſum by the whole number of terms, or the whole ſum by half the number of terms, or laſtly, by multiplying the whole ſum into the whole number of terms and then taking half the product:* thus in the ſeries 2, 4,

6, 8, 10, 12, where the leaſt term is 2, the greateſt 12, their ſum 14, and the number of terms 6; the ſum of all the terms taken together will be 7×6, or 14×3, or $\frac{14\times6}{2}=42$. This will beſt appear by writing down the ſeries 2, 4, 6, 8, 10, 12, and then by writing down over it the ſame ſeries inverted, 12, 10, 8, 6, 4, 2: for if this be done, 2, the firſt term of the lower ſeries added to 12, the firſt term of the upper ſeries, (which is the ſame as the greateſt and leaſt terms of the ſame ſeries added together) will make 14; in like manner every term of the lower ſeries added to the next above it will make 14; therefore both the ſerieſes together will be equal to 14 as often taken as there are terms in either ſeries, that is, 6 times 14, or 84; therefore either ſeries taken alone will be equal to 42.

12	10	8	6	4	2
2.	4.	6.	8.	10.	12.
14.	14.	14.	14.	14.	14.

The deſign of this lemma is, to add the terms of a ſeries together, where only the greateſt and leaſt terms and the number of terms are known, or ſuppoſed to be known; the intermediate terms being either not aſſigned, or too many to be ſummed up by a continual addition.

PROBLEM 82.

125. *A traveller, as* A, *ſets out from a certain place, and travels one mile the firſt day, two miles the ſecond day, three the third, four the fourth*, &c; *and five days after, another, as* B, *ſets out from the ſame place, and travels the ſame road at the rate of twelve miles every day: I demand how long and how far* A *muſt travel before he is overtaken by* B.

SOLUTION.

Put x for the number of days A travelled before he was overtaken by B; then to find an expreſſion for the number of miles travelled by him in that time, I obſerve that in three days A travelled over $1+2+3$ miles, that is, he travels over a ſeries of miles in arithmetical progreſſion, whereof the number of terms is 3, the greateſt term 3, and the leaſt term 1; in four days he travels over a ſeries whereof the number of terms is 4, the greateſt term 4, and the leaſt 1; therefore univerſally, in any number x of days, he muſt travel over a ſeries of miles in arithmetical progreſſion, whereof the number of terms is x, the greateſt term x, and the leaſt term 1; but the ſum of the extremes of this ſeries is $x+1$, which multiplied by x the number of terms, gives $xx+x$, the half where-

whereof is $\frac{xx+x}{2}$; therefore by the lemma foregoing, $\frac{xx+x}{2}$ will be the ſum of this ſeries, and conſequently the miles travelled by A before he was overtaken: again, if A travels x days, B muſt have travelled $x-5$ days, which at the rate of 12 miles a day, gives $12x-60$ for the miles travelled by B when he overtook A; but as they both ſet out from the ſame place, and are now got together, they muſt have travelled the ſame number of miles; whence we have this equation, $\frac{xx+x}{2}=12x-60$; therefore $xx+x=24x-120$; therefore $xx=23x-120$; compare this equation with the general one in art. 103, and you will have $A=1$, $B=23$, $C=-120$, $BB=529$, $4AC=-480$, $ss=49$, $s=7$, $\frac{B+s}{2A}=15$, $\frac{B-s}{2A}=8$; therefore $x=8$, or 15: now for the better application of theſe roots to the ſolution of this problem, it muſt be obſerved, that the problem is more limited than the equation deduced from it; juſt as if, in tranſlating out of one language into another, the terms of the latter, inſtead of being adequate to thoſe of the former, ſhould be found to be of a more extenſive ſignification: in the problem it is only ſuppoſed that B overtakes A, whereas in the equation it is ſuppoſed that A and B are got both together by having travelled the ſame number of miles from their firſt ſetting out, without ſpecifying whether this ariſes from B's overtaking A, or from A's overtaking B; both which in this caſe muſt neceſſarily happen in the courſe of their travels, provided they be but continued long enough for that purpoſe: for ſince at firſt, B is the ſwifter traveller, whenever they come together, it muſt ariſe from B's overtaking A, which happens after A has travelled 8 days; then if we ſuppoſe them ſtill to continue their travels, B paſſes by A, and continues before him for ſome time; but after 12 days, A becomes the ſwifter traveller, and muſt neceſſarily come up to B again after he has travelled 15 days: therefore though the two roots, 8 and 15, will both anſwer the condition of the equation, yet but one of them, to wit, 8, will anſwer the condition of the problem; and that both of them will anſwer the condition of the equation, will be evident as follows.

In 8 days A travels over a ſeries of miles whereof the number of terms is 8, the greateſt 8, and the leaſt 1; the ſum of which ſeries is 36 miles; but when A has travelled 8 days, B muſt have travelled 3 days, during which time, at the rate of 12 miles a day, he alſo muſt have travelled 36 miles; therefore after A had travelled 8 days, A and B muſt neceſſarily find themſelves together: again, in 15 days, A muſt have travel-

travelled over a ſeries of miles, whereof the number of terms is 15, the greateſt 15, the leaſt 1, and the ſum 120 miles; but when *A* had travelled 15 days, *B* muſt have travelled 10 days, which at 12 miles a day gives alſo 120 miles; therefore now again *A* and *B* muſt find themſelves together; and conſequently 8 and 15 equally anſwer the ſuppoſition contained in the equation.

N. B. If we ſuppoſe *B* after 5 days to have begun to ſollow *A*, and to have travelled only 10 miles a day, he could never have overtaken *A*, nor *A* him, ſo that in this caſe, both the roots would have become impoſſible, as will be found by the reſolution of an equation founded upon this ſuppoſition.

PROBLEM 83.

126. *It is required to divide the number ten into two ſuch parts, that the product of their multiplication being added to the ſum of their ſquares, may make ſeventyſix.*

SOLUTION.

The two parts ſought, x and $10-x$.
The product of their multiplication, $10x-xx$.
The ſum of their ſquares, $2xx-20x+100$.
The product of their multiplication added to the ſum of their ſquares, $x^2-10x+100=76$.

Whence $x=4$, or 6; but this equation will be the ſame, which part ſoever x is put for; therefore the two parts ſought are 4 and 6.

PROBLEM 84.

127. *It is required to find two numbers with the following properties, to wit, that twice the firſt with three times the ſecond may make ſixty, and moreover, that twice the ſquare of the firſt with three times the ſquare of the ſecond may make eight hundred and forty.*

SOLUTION.

For the two numbers ſought put x and y, and we ſhall have

Equ. 1ſt, $2x+3y=60$, and

Equ. 2d, $2x^2+3y^2=840$.

From the firſt equation, $2x+3y=60$, we have

Equ. 3d, $x=\frac{60-3y}{2}$; and by ſquaring both ſides we have

Equ. 4th, $xx=\frac{3600-360y+9yy}{4}$.

From

From the ſecond equation, $2xx+3yy=840$, we have

$$\text{Equ. 5th, } xx=\frac{840-3yy}{2}.$$

Compare the two values of xx in the fourth and fifth equations, which muſt neceſſarily be equal one to the other, and you will have $\frac{3600-360y+9yy}{4}=\frac{840-3yy}{2}$; multiply both ſides into 2, by halving the denominators, and you will have $\frac{3600-360y+9yy}{2}=840-3yy$; therefore $3600-360y+9yy=1680-6yy$; therefore $3600-360y+15yy=1680$; therefore $15yy-360y=-1920$; therefore $15yy=360y-1920$; divide by 15 for a more ſimple equation, and you will have $yy=24y-128$; whence $y=8$, or 16: ſuppoſe $y=8$, then ſince by the third equation $x=\frac{60-3y}{2}$, we ſhall have $x=18$; ſuppoſe $y=16$, then we ſhall have x or $\frac{60-3y}{2}=6$; therefore there are two pair of numbers that will equally anſwer the conditions of this problem, to wit, 18 and 8, and alſo 6 and 16: for a proof, let us firſt ſuppoſe the numbers to be 18 and 8; and we ſhall have twice the firſt number with three times the ſecond $=36+24=60$; and twice the ſquare of the firſt together with three times the ſquare of the ſecond equal to $648+192=840$: ſecondly, let us ſuppoſe the numbers to be 6 and 16; and we ſhall have twice the firſt with three times the ſecond equal to $12+48=60$; and twice the ſquare of the firſt with three times the ſquare of the ſecond equal to $72+768=840$.

PROBLEM 85.

128. *To find four numbers in continual proportion, and ſuch, that the ſum of the two middle terms may be eighteen, and that of the extremes twentyſeven.*

Note, Four numbers are ſaid to be in continual proportion, when the firſt is to the ſecond as the ſecond is to the third, and the ſecond is to the third as the third is to the fourth.

SOLUTION.

For the two middle terms put x and y, without intending which is to be the greater; then the extreme next to x may be found by ſaying, as y is to x ſo is x to $\frac{xx}{y}$, and the extreme next to y may be found by

ſaying,

saying, as x is to y so is y to $\frac{yy}{x}$; therefore the extremes are $\frac{xx}{y}$ and $\frac{yy}{x}$, and their sum $\frac{x^3+y^3}{xy}$; therefore the fundamental equations are 1st, $x+y=18$, or $x=18-y$; and 2dly, $\frac{x^3+y^3}{xy}=27$, or $x^3+y^3=27xy$; instead of x in this equation put $18-y$, it's value in the last, and you will have $x^3=5832-972y+54y^2-y^3$; therefore $x^3+y^3=5832-972y+54yy$; you will also have $27xy$ or $27y\times\overline{18-y}=486y-27yy$; therefore $5832-972y+54yy=486y-27yy$; transpose $486y-27yy$, and you will have $81yy-1458y+5832=0$; divide all by 81, which may be done without a fraction, and you will have $yy-18y+72=0$; which equation being resolved, either by the general theorem or any other way, gives $y=6$, or 12; and since the equation will be the same, whichsoever of the two middle terms y stands for, it follows, that the two middle terms are 6 and 12; whence the extreme next to 6 is 3, and that next to 12 is 24; and the numbers are either 3, 6, 12 and 24, or 24, 12, 6 and 3, for either way they will answer the conditions of the problem.

PROBLEM 86.

129. *There are three numbers in continual proportion, whose sum is nineteen, and the sum of their squares one hundred thirtythree: What are the numbers?*

SOLUTION.

For the three numbers sought put x, y and z; then since by the first condition, x is to y as y is to z, by multiplying extremes and means we have $yy=xz$; again, by the second condition of the problem we have $x+y+z=19$, and $19-y=x+z$, and (squaring both sides) $361-38y+yy=xx+2xz+zz$; subtract yy from one side of the equation, and it's equal xz from the other, and you will have $361-38y=x^2+xz+z^2=x^2+y^2+z^2=133$ by the third condition of the problem: having thus expunged both x and z at once, resolve the equation $361-38y=133$, and you will have y the middle term equal to 6, and $19-y$, or the sum of the extremes $=13$; therefore the problem proposed is now reduced to this, *viz. Of three numbers in continual proportion, whereof six the middle term, and thirteen the sum of the extremes are given, to find the extremes:* this problem is of the same nature with that in art. 112, and being resolved, gives 4 and 9 for the extremes; therefore the three numbers sought are 4, 6 and 9, or 9, 6 and 4.

PRO-

PROBLEM 87.

130. *To find two numbers ſuch, that their difference multiplied into the difference of their ſquares ſhall make thirtytwo, but their ſum multiplied into the ſum of their ſquares ſhall make two hundred ſeventytwo.*

SOLUTION.

For the two numbers ſought put x and y; and the firſt fundamental equation will be $\overline{x-y}\times\overline{x^2-y^2}$, or $\overline{x-y}\times\overline{x-y}\times\overline{x+y}$, or $\overline{x^2-2xy+y^2}\times\overline{x+y}=32$; therefore

Equ. 1ſt, $x^2-2xy+y^2=\frac{32}{x+y}$.

The ſecond fundamental equation is, $\overline{x+y}\times\overline{x^2+y^2}=272$; therefore

Equ. 2d, $x^2+y^2=\frac{272}{x+y}$.

From twice the ſecond equation ſubtract the firſt, that is, from $2x^2 \quad * + 2y^2=\frac{544}{x+y}$

ſubtract $x^2-2xy+y^2=\frac{32}{x+y}$

and you will have $x^2+2xy+y^2=\frac{512}{x+y}$,

that is, $\overline{x+y}^2=\frac{512}{x+y}$; therefore $\overline{x+y}^3=512$, and $x+y=\sqrt[3]{512}$, or the cube root of $512=8$: thus we have got the ſum of the two numbers ſought, to wit, 8; whence their difference may be found by the firſt equation, thus; $x^2-2xy+yy=\frac{32}{x+y}$, that is, $\overline{x-y}^2=\frac{32}{8}=4$; therefore $x-y$, or the difference of the two numbers ſought, equals 2; therefore the problem propoſed is now reduced to this; *Having given eight the ſum, and two the difference of the two numbers* x *and* y, *to find thoſe numbers*; and by art. 26 we ſhall have $x=5$, and $y=3$; which numbers will anſwer the conditions of the queſtion.

N. B. After we had found $x+y$, the ſum of the numbers equal to 8, we might have found the ſum of their ſquares by the ſecond equation, which gave $x^2+y^2=\frac{272}{x+y}=\frac{272}{8}=34$; and then the problem would have been reduced to this; *What two numbers are thoſe, whoſe ſum is eight, and the ſum of their ſquares thirtyfour?* which would have produced a quadratic equation, as in art. 113, whoſe two roots would have been 5 and 3, as before.

PROBLEM 88.

131. *To find two numbers such, that their difference added to the difference of their squares may make fourteen, and their sum added to the sum of their squares may make twenty six.*

SOLUTION.

For the two numbers sought put x and y, and you will have the two following equations;

Equ. 1st, $x-y+x^2-y^2=14$.
Equ. 2d, $x+y+x^2+y^2=26$.

Add these two equations together, and you will have $2xx+2x=40$, $xx+x=20$, and $x=+4$, or -5; again, subtract the first equation from the second, and you will have $2yy+2y=12$, $yy+y=6$, and $y=+2$, or -3; and as these two values of y were obtained without any manner of dependence upon those of x, it is plain that either of the values of x may be joined with either of the values of y; and so we have no fewer than four pairs of numbers which will equally satisfy the conditions of the equations, to wit, $+4$ and $+2$, $+4$ and -3, -5 and $+2$, -5 and -3; but it is the first pair only, which, consisting of affirmative numbers, is proper for the solution of the problem, thus; the difference of 4 and 2 is 2, the difference of their squares 12, and $2+12=14$; again, the sum of 4 and 2 is 6, the sum of their squares 20, and $6+20=26$: let us see however how the other pairs will satisfy the conditions of the equations; make then x equal to 4, y, that is, $+y=-3$, and you will have $-y=+3$; whence $x-y=4+3=7$, $x^2-y^2=16-9=7$, and $7+7=14$; again, $x+y=4-3=1$, and $x^2+y^2=16+9=25$, and $1+25=26$: in the next place, make $x=-5$, and $y=+2$, then we shall have $x-y=-5-2=-7$, $x^2-y^2=25-4=21$, and $-7+21=14$; again, $x+y=-5+2=-3$, and $x^2+y^2=25+4=29$, and $-3+29=26$: lastly, make $x=-5$, and $y=-3$, and you will have $x-y=-5+3=-2$, and $x^2-y^2=25-9=16$, and $-2+16=14$; again, $x+y=-5-3=-8$, and $x^2+y^2=25+9=34$, and $-8+34=26$.

PROBLEM 89.

132. *What two numbers are those, whose sum, when added together, is equal to their product when multiplied together; and this sum or product, when added to the sum of their squares, makes twelve?*

For

SOLUTION.

For the two numbers ſought put x and y, and the fundamental equations will be 1ſt, $x+y=xy$; and ſecondly, $x+y+x^2+y^2=12$: in the firſt of theſe fundamental equations, where $x+y=yx$, we have $yx-x=y$; but $yx-x$ is the product of $\overline{y-1}\times x$, or of $x\times\overline{y-1}$; therefore $x\times\overline{y-1}=y$, and $x=\frac{y}{y-1}$; but if inſtead of x, this value be ſubſtituted into the ſecond fundamental equation, the equation will riſe to a biquadratic, for the reſolution whereof, no rules have hitherto been given; therefore to extricate ourſelves out of this difficulty, it will be proper to have recourſe to ſome other artifice, by trying other poſitions, as thus; for the ſum of the two numbers ſought put z; then will z be alſo the product of their multiplication, by the ſuppoſition; and ſince this product z added to the ſum of their ſquares gives 12, the ſum of their ſquares will be $12-z$; but every one knows, that if to the ſum of the ſquares of any two numbers be added their double product, there will ariſe the ſquare of their ſum; therefore $12-z+2z$, or $12+z=z^2$; which equation being reſolved, gives $z=+4$, &c; and therefore the queſtion is now reduced to this; *What two numbers are thoſe, whoſe ſum is four, and the product of whoſe multiplication is four?* for the numbers ſought, put x and $4-x$, and you will have $4x-xx=4$; and changing the ſigns, $xx-4x=-4$; and compleating the ſquare, $xx-4x+4=0$; and extracting the ſquare root, $x-2=\pm 0$; whence $x=2$, or 2, for the roots of this equation are equal; therefore 2 and 2 are the numbers deſired in the queſtion; and they will anſwer the conditions; for in the firſt place, $2+2=4=2\times 2$; and in the next place, 4 the ſum of 2 and 2, being added to 8, the ſum of their ſquares, gives 12.

COROLLARY.

From our firſt attempt to ſolve this problem we may learn thus much however, that if any number whatever be made equal to y, then theſe two numbers y and $\frac{y}{y-1}$ will always have this property, that their ſum when added together will be equal to their product when multiplied together; thus if $3=y$, and conſequently $\frac{3}{2}=\frac{y}{y-1}$, we ſhall have $3+\frac{3}{2}=4\frac{1}{2}$, and $3\times\frac{3}{2}$ or $\frac{9}{2}=4\frac{1}{2}$; whence it follows, that this problem cannot be ſolved in whole numbers in any other caſe than that we have here put.

PROBLEM 90.

133. *What two numbers are thoſe, whoſe ſum added to the product of their multiplication makes thirtyfour, and the ſame ſum ſubtracted from the ſum of their ſquares leaves fortytwo?*

SOLUTION.

Here to avoid all difficulties that would otherwiſe ariſe, put z for the ſum of the two numbers ſought; then ſince this ſum added to the product of their multiplication makes 34, the product of their multiplication will be $34 - z$; but this ſum z ſubtracted from the ſum of their ſquares, leaves 42; therefore the ſum of their ſquares is $42 + z$; to this add their double product $68 - 2z$, and you will have $110 - z = z^2$; whence $z = + 10$, &c, and $34 - z = 24$; therefore now the queſtion is, *What two numbers are thoſe, whoſe ſum is ten, and the product of their multiplication twentyfour?* and by art. 111 the two numbers ſought are 4 and 6.

Whoever would ſee more queſtions of this nature, may conſult *Bachet*'s comment upon the 33d queſtion of the firſt book of *Diophantus*'s Arithmetics.

N. B. Having now done with quadratic equations, at leaſt for a time, it may perhaps be expected, that according to order of method I ſhould proceed on to equations of higher forms: but I ſhall take the liberty for once to diſpence with that method; not but that I intend (God willing) to treat fully and diſtinctly of theſe equations hereafter; but in the mean time I think it more adviſeable to employ the reader's thoughts in ſome other things, which I take to be of much greater importance, and more proper for his information.

THE

ELEMENTS of ALGEBRA

BOOK IV

Of general problems, and general theorems deduced from them; together with the manner of applying, and demonſtrating theſe theorems ſynthetically.

The deſign of this fourth book more fully explained.

134. HITHERTO my young Analyſt has been indulged for the moſt part in a ſort of mixt Algebra, where letters were put only for unknown quantities: but if he would reaſon abſtractedly upon his problems, and draw general concluſions from them, he muſt put letters not only for his unknown quantities, but alſo for ſuch as are known; and ſo propoſe and ſolve his problems indefinitely. By this means in the firſt place he will obtain indefinite anſwers, which in many caſes are much preferable to more particular ones, as they ſuit and ſolve all particular caſes to which they are applicable; and in the next place he will be able to prove his work ſynthetically; which will not only confirm his former *analyſis*, but will alſo further enure and reconcile him to the operations of ſymbolical or ſpecious Arithmetic; and ſo render him entire maſter of this ſort of computation. A ſufficient ſpecimen of this ſort of reaſoning both in the analytical and ſynthetical way, has already been given in our general theorem for the reſolution of a quadratic equation, ſo that no more needs be ſaid by way of preparation; it remains therefore now, that we look back upon ſome of the problems already ſolved, and ſhew how to ſolve them over again in general terms, as follows.

PROBLEM I. (See art. 26.)

135. *What two numbers are those, whose sum is* a, *and difference* b?

SOLUTION.

Put x for the less number; then will the greater be $x+b$, and their sum $2x+b=a$; whence $2x=a-b$, and x (the less number) will be $\frac{a-b}{2}$; whence $x+b$ (the greater number) will be $\frac{a-b}{2}+\frac{b}{1}=\frac{a-b+2b}{2}=\frac{a+b}{2}$; so the greater number is found to be $\frac{a+b}{2}$, and the less $\frac{a-b}{2}$; where a and b are left undetermined till some particular case of this problem is proposed to be compared with the general one; and then the quantities a and b will not only be determined in that case, but the problem may be solved by the general theorem without any further *analysis*. As for example, let it be proposed, as in art. 26, to find two numbers whose sum is 48, and difference 14: here it is plain that a in the general problem answers to 48 in the particular case, and b to 14; whence $\frac{a+b}{2}$ (or the greater number) $=\frac{48+14}{2}=\frac{62}{2}=31$, and $\frac{a-b}{2}$ (or the less number) $=\frac{48-14}{2}=\frac{34}{2}=17$; so that the numbers sought are 31 and 17; which will answer the conditions of the question. Again, suppose we were to find two numbers whose sum is 35, and whose difference is 9: in this case it is plain that a and b have other significations; for here $a=35$, and $b=9$, and therefore $\frac{a+b}{2}$ (or the greater number) will be 22, and $\frac{a-b}{2}$ (or the less number) will be 13.

These theorems are capable of being translated out of Algebraic language into any other; though to no great purpose that I know of, to such as understand any thing of symbolical Arithmetic; for in my opinion, they appear much more distinct as they are, and less liable to ambiguity. The foregoing problem, together with the answer belonging to it, being translated into common English, will stand thus:

PROBLEM.

It is required, having given the sum and difference of any two numbers, to find the numbers themselves.

Anf. 1ft, *Add the difference to the fum, and half the aggregate will be the greater number.* 2dly, *Subtract the difference from the fum, and half the remainder will be the lefs number.*

That this is a true tranflation is plain: for what is $\frac{a+b}{2}$ but half the aggregate of the fum and difference added together? and what is $\frac{a-b}{2}$ but half the remainder, after the difference is fubtracted from the fum?

We come now in the laft place to examine this theorem as it ftands in general terms, and to try whether it will anfwer the conditions of the problem in the letters themfelves. It was propofed to find two numbers, whofe fum is a, and whofe difference is b; and the anfwer was, that the greater number was $\frac{a+b}{2}$, and the lefs $\frac{a-b}{2}$: now that this is a true anfwer, will be evident from a bare addition and fubtraction of the numbers themfelves, without any other principles; for if $\frac{a+b}{2}$ be added to $\frac{a-b}{2}$, their fum will be $\frac{2a}{2}$ or a, which anfwers the firft condition of the problem; and if $\frac{a-b}{2}$ be fubtracted from $\frac{a+b}{2}$, the remainder will be $\frac{2b}{2}$ or b, which anfwers the fecond condition.

This is that which is called a fynthetical demonftration, and doubtlefs fhews the truth of the theorem to which it belongs, as well as the *analyfis* whereby that theorem was inveftigated; but not fo much to the fatiffaction of the mind: for a fynthetical demonftration only fhews that a propofition is true; whereas an analytical one fhews not only that a propofition is true, but why it is fo, places you in the condition of the inventer himfelf, and unveils the whole myftery. Synthetical demonftrations ufually require fewer principles than analytical ones, as will evidently appear, by comparing both, in this very example; and this I take to be the reafon why the ancients, generally fpeaking, chofe to demonftrate their propofitions this way; not with a defign to conceal their *analyfis*, as fome have, unjuftly enough, imagined; but becaufe this fort of demonftration required fewer principles to proceed upon, and thofe too, fuch as were commonly known.

PROBLEM 2.

136. *What three numbers are thofe, whereof the fum of t[illegible] fecond is* a, *that of the firft and third* b, *and that of the fecond and third* c?

SOLU-

SOLUTION.

Put x for the firſt number ſought; then will the ſecond number be $a-x$, becauſe the firſt and ſecond numbers together make a; for a like reaſon the third number will be $b-x$, becauſe the firſt and third together make b: add now the ſecond and third numbers together, and you will have $a+b-2x=c$; therefore $2x+c=a+b$; therefore $2x=a+b-c$; and x (or the firſt number) $=\frac{a+b-c}{2}$; ſubtract now this firſt number $\frac{a+b-c}{2}$ from a, or which is all one, add $\frac{-a-b+c}{2}$ to a, and you will have the ſecond number equal to $\frac{-a-b+c}{2}+\frac{a}{1}=\frac{-a-b+c+2a}{2}=\frac{a-b+c}{2}$; again, ſubtract the firſt number $\frac{a+b-c}{2}$ from b, and you will have the third number equal to $\frac{-a-b+c}{2}+\frac{b}{1}=\frac{-a+b+c}{2}$; and thus we have all the three numbers ſought, to wit,

The firſt, $\frac{a+b-c}{2}$,

The ſecond, $\frac{a-b+c}{2}$,

The third, $\frac{-a+b+c}{2}$.

To apply this general ſolution to ſome particular caſe, I ſhall make uſe of that in art. 42, where it was required to find three ſuch numbers, that the ſum of the firſt and ſecond may make 60, that of the firſt and third 80, and that of the ſecond and third 92: in this caſe it is plain that $a=60$, $b=80$, and $c=92$; therefore $\frac{a+b-c}{2}$ or the firſt number will be 24; $\frac{a-b+c}{2}$ or the ſecond number will be 36; and $\frac{-a+b+c}{2}$ or the third number will be 56; which numbers upon tryal will be found to be ſuch as the problem requires. But that the theorems here given are not only true in this particular caſe, but are univerſally ſo, will beſt appear from the ſynthetical demonſtration following.

1ſt, The firſt number $\frac{a+b-c}{2}$, and the ſecond number $\frac{a-b+c}{2}$ being added together make $\frac{2a}{2}$ or a, according to the the firſt condition, the other quantities deſtroying one another.

2dly, The firſt number $\frac{a+b-c}{2}$, and the third number $\frac{-a+b+c}{2}$ being added together make $\frac{2b}{2}$ or b, according to the ſecond condition.

Laſtly, The ſecond number $\frac{a-b+c}{2}$ and the third number $\frac{-a+b+c}{2}$ being added together make $\frac{2c}{2}$ or c, according to the third condition.

This problem may alſo be ſolved ſomewhat more elegantly thus: put s for the unknown ſum of all the three numbers ſought; then if c, the ſum of the ſecond and third numbers be ſubtracted from s, the ſum of all three, there will remain the firſt number equal to $s-c$; in like manner b, the ſum of the firſt and third numbers, ſubſtracted from s, the ſum of all three, leaves the ſecond number equal to $s-b$; and a, the ſum of the firſt and ſecond numbers, ſubtracted from s, the ſum of all three, leaves the third number equal to $s-a$; add now all theſe three numbers together, to wit, $s-c$, $s-b$ and $s-a$, and the ſum will be $3s-a-b-c$; but the ſum is s, by the ſuppoſition; therefore $3s-a-b-c=s$; and $s=\frac{a+b+c}{2}$; whence we have the following theorem:

Make $\frac{a+b+c}{2}=s$; *then if the numbers* a, b *and* c *be taken backwards, and ſubtracted ſeverally from* s, *the three remainders* $\overline{s-c}$, $\overline{s-b}$ *and* $\overline{s-a}$ *will be the three numbers ſought, in order as they are ſuppoſed in the problem.* Thus if $a=60$, $b=80$, and $c=92$, as before, we ſhall have $\frac{a+b+c}{2}$ or $s=116$; whence the firſt number will be $116-92$ or 24, the ſecond $116-80$ or 36, and the third $116-60$ or 56.

SCHOLIUM.

What three numbers are thoſe, whereof the product of the firſt and ſecond is a, *that of the firſt and third* b, *and that of the ſecond and third* c?

SOLUTION.

Put p for the product of all the three numbers; then since c is the product of the two last, we shall have the first number equal to $\frac{p}{c}$; for a like reason the second equals $\frac{p}{b}$, and the third equals $\frac{p}{a}$, and the product of all three equals $\frac{p^3}{abc}=p$; therefore $p^2=abc$, and $p=\sqrt{abc}$.

DEMONSTRATION.

$\frac{p}{c}\times\frac{p}{b}$, or the product of the first and second numbers, is $\frac{p^2}{bc}=\frac{abc}{bc}=a$; and so of the rest.

PROBLEM 3.

137. *It is required to find two numbers whose difference is* b, *and the difference of whose squares is* a.

SOLUTION.

Put x for the less number, and consequently $x+b$ for the greater; then will the square of the less number be xx, that of the greater $xx+2bx+bb$, and the difference of their squares $2bx+bb=a$; therefore $2bx=a-bb$, and x (the less number) $=\frac{a-bb}{2b}$; whence $x+b$ (the greater) $=\frac{a-bb}{2b}+\frac{b}{1}=\frac{a-bb+2bb}{2b}=\frac{a+bb}{2b}$.

To apply this general solution, let it be required to find two numbers whose difference is 4, and the difference of whose squares is 112: here $a=112$, $b=4$, $bb=16$, $\frac{a-bb}{2b}=12$, $\frac{a+bb}{2b}=16$; therefore the numbers are 12 and 16. The general demonstration is as follows: if the less number $\frac{a-bb}{2b}$ be subtracted from the greater $\frac{a+bb}{2b}$, their difference will be $\frac{2bb}{2b}$ or b, according to the first condition of the problem; again, the square of the less number $\frac{a-bb}{2b}$ is $\frac{aa-2abb+b^4}{4bb}$, and the square of the greater $\frac{a+bb}{2b}$ is $\frac{aa+2ab^2+b^4}{4bb}$; subtract the square of the less

leſs from that of the greater, and you will have the difference of their ſquares $= \frac{4abb}{4bb} = a$, as the ſecond condition requires.

PROBLEM 4.

138. *Let* r *and* s *be two given multiplicators, whereof* r *is the greater; it is required to divide a given number as* a *into two ſuch parts, that the greater part when multiplied into the leſs multiplicator may be equal to the leſs part when multiplied by the greater multiplicator.*

SOLUTION.

Put x for the greater part, and $a-x$ for the leſs; then will the greater part multiplied into the leſs multiplicator be sx, and the leſs part multiplied into the greater multiplicator will be $ar-rx$; but according to the problem, theſe products are to be equal; therefore $sx=ar-rx$, and $rx+sx=ar$; but $rx+sx$ is $x\times\overline{r+s}$; therefore $x\times\overline{r+s}=ar$; and x (the greater of the two parts ſought) $=\frac{ar}{r+s}$; whence $a-x$, (the leſs part) equals $\frac{a}{1}-\frac{ar}{r+s}=\frac{ar+as-ar}{r+s}=\frac{as}{r+s}$; ſo the greater part ſought is $\frac{ar}{r+s}$, and the leſs $\frac{as}{r+s}$.

The APPLICATION.

To apply this canon, let it be required to divide 84 into two ſuch parts, that five times one part may be equal to ſeven times the other: here $a=84$, r the greater multiplicator $=7$, $s=5$, $\frac{ar}{r+s}=\frac{7\times84}{12}=49$, $\frac{as}{r+s}=\frac{5\times84}{12}=35$; therefore the greater part is 49, and the leſs 35; and they will anſwer the conditions; for firſt, $49+35=84$; and ſecondly, $49\times5=245=35\times7$. Again, let it be required to divide 99 into two ſuch parts, that $\frac{2}{3}$ of one part may be equal to $\frac{4}{5}$ of the other: here $a=99$, $r=\frac{4}{5}$, $s=\frac{2}{3}$, $r+s=\frac{22}{15}$, $\frac{r}{r+s}=\frac{\frac{4}{5}}{\frac{22}{15}}=\frac{6}{11}$, $\frac{s}{r+s}=\frac{\frac{2}{3}}{\frac{22}{15}}=\frac{5}{11}$, $\frac{ar}{r+s}=99\times\frac{6}{11}=54$, $\frac{as}{r+s}=99\times\frac{5}{11}=45$; ſo the two parts are 54 and 45, which is true; for firſt, $54+45=99$; and ſecondly, $\frac{2}{3}$ of $54=$ $=\frac{4}{5}$ of 45.

As to the demonſtration of this general ſolution, it muſt be obſerved that in this problem there are two conditions; firſt, that the two parts when added together muſt make a; and ſecondly, that the greater part multiplied into the leſs multiplicator muſt be equal to the leſs part multiplied into the greater multiplicator: as to the firſt of the conditions, it is certain that the parts $\frac{ar}{r+s}$ and $\frac{as}{r+s}$ when added together will make $\frac{ar+as}{r+s}$; but $ar+as=a\times\overline{r+s}$, therefore $\frac{ar+as}{r+s}=a\times\frac{r+s}{r+s}=a\times 1=a$: as to the ſecond condition, if the greater part $\frac{ar}{r+s}$ be multiplied into s, the leſs multiplicator, the product will be $\frac{ars}{r+s}$; and again, if the leſs part $\frac{as}{r+s}$ be multiplied into r, the greater multiplicator, the product will alſo be $\frac{ars}{r+s}$; therefore the two products are equal, as the problem requires; and ſo the conditions are both ſatisfied. *Q. E. D.*

N. B. If any one has a mind to throw the foregoing theorem into words, it may eaſily be done, and in ſuch a manner as almoſt to carry it's own evidence along with it; for by the rule of proportion, $r+s$ is to r as a to $\frac{ar}{r+s}$; and $r+s$ is to s as a to $\frac{as}{r+s}$; therefore, *As the ſum of the two multiplicators is to the greater or leſs multiplicator, ſo is the ſum of the two parts ſought to the greater or leſs part:* and this, I ſay, is pretty evident; for had $\overline{r+s}$ been the number to be divided, the parts would certainly have been r and s; therefore if a greater or leſs number than $\overline{r+s}$ is to be divided, the parts ought to be greater or leſs than r and s in the ſame proportion.

PROBLEM 5.

139. *Let* r *and* s *be two given multiplicators whereof* r *is the greater; it is required to divide a given number as* a *into two ſuch parts, that* r *times one part being added to* s *times the other may make ſome other given number, as* b.

SOLUTION.

Put x for the part that is to be multiplied by r, and conſequ $a-x$ for the other part that is to be multiplied by s, and the pro

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will be rx and $as-sx$, and their ſum will be $rx+as-sx=b$; therefore $rx-sx=b-as$, that is, $x\times\overline{r-s}=b-as$; therefore x (the part to be multiplied by r) $=\frac{b-as}{r-s}$; therefore $a-x$ (the part to be multiplied by s) $=\frac{a}{1}-\frac{b+as}{r-s}=\frac{ar-as-b+as}{r-s}=\frac{ar-b}{r-s}$.

The APPLICATION.

Let it be required to divide 20 into two ſuch parts, that three times one part being added to five times the other may make 84: here $a=20$, $b=84$, $r=5$, $s=3$, $as=60$, $b-as=24$, $\frac{b-as}{r-s}$ (or the part to be multiplied by 5) $=\frac{24}{2}=12$, $ar=100$, $ar-b=16$, $\frac{ar-b}{r-s}$ (or the part to be multiplied by 3) $=\frac{16}{2}=8$; therefore the parts ſought are 8 and 12; for firſt, $8+12=20$; and ſecondly, three times 8 + five times $12=84$.

Again, let it be required to divide 100 into two ſuch parts, that $\frac{3}{4}$ of one part being ſubtracted from $\frac{5}{6}$ of the other, may leave 39: here it muſt be obſerved, that to ſubtract $\frac{3}{4}$ of any one quantity from another, is the ſame as to add $\frac{-3}{4}$ of it; therefore this problem when reduced to the form of the general one, will ſtand thus: *To divide a hundred into two ſuch parts, that* $\frac{-3}{4}$ *of one part being added to* $+\frac{5}{6}$ *of the other may make thirtynine.* Here $a=100$, $b=39$, $r=\frac{5}{6}$, $s=\frac{-3}{4}$, $r-s=\frac{5}{6}+\frac{3}{4}=\frac{19}{12}$, $as=\frac{-300}{4}=-75$, $b-as=39+75=114$, $\frac{b-as}{r-s}=\frac{114}{\frac{19}{12}}=72$, $ar=\frac{500}{6}=\frac{250}{3}$, $ar-b=\frac{250}{3}-\frac{39}{1}=\frac{133}{3}$, $\frac{ar-b}{r-s}=\frac{\frac{133}{3}}{\frac{19}{12}}=28$; ſo the two parts are 28 and 72; for $28+72=100$; and moreover $\frac{3}{4}$ of 28, that is, 21, ſubtracted from $\frac{5}{6}$ of 72, that is, from 60, leaves 39.

The GENERAL DEMONSTRATION.

The two parts $\frac{ar-b}{r-s}$ and $\frac{b-as}{r-s}$ when added together, make

$ar-b$

$\frac{ar-b+b-as}{r-s}=\frac{ar-as}{r-s}=a\times\frac{r-s}{r-s}=a$: again, the part $\frac{b-as}{r-s}$ being multiplied into r, it's proper multiplicator, gives $\frac{br-ars}{r-s}$, and the other part $\frac{ar-b}{r-s}$, multiplied into the other multiplicator s, gives $\frac{ars-bs}{r-s}$; add these two products together, and they will make $\frac{br-ars+ars-bs}{r-s}=\frac{br-bs}{r-s}=b$. *Q. E. D.*

If any one hereafter shall think me too concise in the solutions of these general problems, he must have recourse to the particular ones in the articles I shall refer him to, which he will find explained more at large: and as to the application of these general solutions to those particular cases, it is to be presumed that by this time the learner will be able in some measure to perform that part himself; and therefore I shall for the future leave it to him, except where I shall think my assistance may be of any use.

PROBLEM 6. (See art. 35.)

140. *One meeting a company of beggars, gives to each* p *pence, and has* a *pence over; but if he would have given them* q *pence apiece, he would have found he had wanted* b *pence for that purpose: What was the number of persons?*

SOLUTION.

The number of persons, x.
Pence given, px.
Pence in all, $px+a$.
The pence that would have been given upon the other supposition, qx.
Another expression for the number of pence in all, $qx-b$.
Equ. $qx-b=px+a$; therefore $qx-px-b=a$; therefore $qx-px=a+b$; therefore x (the number of persons) $=\frac{a+b}{q-p}$.

DEMONSTRATION.

If the number of persons be $\frac{a+b}{q-p}$, then the pence given will $\frac{ap+bp}{q-p}$, and the pence in all will be $\frac{ap+bp}{q-p}+\frac{a}{1}=\frac{ap+bp+aq}{q-p}$

$=\frac{aq+bp}{q-p}$: again, the number of pence that would have been given upon the ſecond ſuppoſition is $\frac{aq+bq}{q-p}$; and therefore the other expreſſion for the number of pence in all will be $\frac{aq+bq}{q-p}-\frac{b}{1}=\frac{aq+bp}{q-p}$; and the perfect agreement between this account and the former, is an infallible argument that the number of perſons was rightly aſſigned.

PROBLEM 7. (See art. 64.)

141. *It is required to divide a given number as* a *into two ſuch parts, that one part may be to the other as* r *to* s.

SOLUTION.

The two parts ſought, x and $a-x$.
Proportion, x is to $a-x$ as r to s.
Equation, $sx=ar-rx$; therefore $rx+sx=ar$; therefore x (or the firſt number) $=\frac{ar}{r+s}$; therefore $a-x$ (or the ſecond number) $=\frac{a}{1}-\frac{ar}{r+s}=\frac{as}{r+s}$; therefore the two numbers are $\frac{ar}{r+s}$ and $\frac{as}{r+s}$.

DEMONSTRATION.

1ſt, The two numbers $\frac{ar}{r+s}$ and $\frac{as}{r+s}$ when added together make $\frac{ar+as}{r+s}=a$.

2dly, The firſt number $\frac{ar}{r+s}$ is to the ſecond number $\frac{as}{r+s}$ as ar is to as; becauſe throwing away the common denominator is no more in reality than multiplying both fractions by it; and every one knows, that the multiplication of two quantities by the ſame number, makes no alteration in the proportion they bore one to the other: again, ar is to as (dividing both by a) as r to s; for it is as well known that a common diviſion affects proportion no more than a common multiplication: ſince then the firſt number is to the ſecond as ar to as, and ar is to as as r to s, it follows, that the firſt number is to the ſecond as r to s. *Q. E. D.*

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PROBLEM 8. (See art. 66.)

142. *What number is that, which being ſeverally added to two given numbers,* a *a greater number, and* b *a leſs, will make the former ſum to the latter as* r *to* s*? therefore* r *muſt be greater than* s.

SOLUTION.

The number ſought, x.

Proportion, $a+x$ is to $b+x$ as r to s.

Equation, $br+rx=as+sx$; therefore $br+rx-sx=as$; therefore $rx-sx=as-br$; therefore $x=\frac{as-br}{r-s}$.

DEMONSTRATION.

The number $\frac{as-br}{r-s}$ being added to a, gives $\frac{ar-br}{r-s}$; and the ſame number being added to b, gives $\frac{as-bs}{r-s}$; now $\frac{ar-br}{r-s}$ is to $\frac{as-bs}{r-s}$ as $ar-br$ is to $as-bs$, that is, as $r\times\overline{a-b}$ is to $s\times\overline{a-b}$, that is, as r to s. *Q. E. D.*

SCHOLIUM.

This problem was to find a number, which being ſeverally added to a and b, will make the former ſum to the latter as r to s; let us now change the numbers a and b one for another, as alſo the numbers r and s one for another, and then the problem will ſtand thus: *To find a number which being ſeverally added to* b *and* a, *will make the former ſum to the latter as* s *to* r: but the condition of this problem is exactly the ſame with that of the former, and therefore the anſwer ought ſtill to be the ſame; that is, as changing a and b one for another, and r and s one for another, had no effect upon the problem, but left it entirely the ſame as at firſt; ſo if the expreſſion of the number ſought be juſt, the changing of a and b one for another, and of r and s one for another, ought to make no alteration in that expreſſion, and the number ſought ought ſtill to be the ſame; for truth will always be conſiſtent with herſelf. Let us try this however, and ſee what will be the effect of ſuch a change: now the number ſought was $\frac{as-br}{r-s}$; but upon this change, as becomes br, and br becomes as, and $r-s$ becomes $s-r$, and the whole expreſſion will be turned into this, $\frac{br-as}{s-r}$; but $\frac{br-as}{s-r}$ is the ſame

as

as $\frac{as-br}{r-s}$; for changing the ſign of both the numerator and denominator of any fraction, no more affects the value of that fraction, than in diviſion the changing of the ſign both of the diviſor and dividend affects the value of the quotient: thus then we find, that the changing of a and b one for another and of r and s one for another, no more affects the theorem for determining the number ſought, than it did the problem from whence it was derived.

PROBLEM 9.

143. *It is required to divide a given number as* a *into two ſuch parts, that the exceſs of one part above another given number as* b, *may be to what the other wants of* b, *as* r *to* s; *ſuppoſing* r *greater than* s.

SOLUTION.

Put x for the greater part, and $\overline{a-x}$ for the leſs; then the exceſs of x above b will be $x-b$; and the exceſs of b above $\overline{a-x}$ will be $x-a+b$, as appears by ſubtracting $\overline{a-x}$ from b; but by the problem, the former exceſs is to the latter as r to s; therefore $x-b$ is to $x-a+b$ as r to s; multiply extremes and means, and you will have $sx-bs=rx-ar+br$; therefore $rx-sx=ar-br-bs$, and x (the greater part) $=\frac{ar-br-bs}{r-s}$; therefore $a-x$ (the leſs part) $=\frac{a}{1}-\frac{ar+br+bs}{r-s}=\frac{br+bs-as}{r-s}$; ſo the greater part is $\frac{ar-br-bs}{r-s}$, and the leſs part $\frac{br+bs-as}{r-s}$.

EXAMPLE.

Let it be required (as in art. 41,) to divide the number 48 into two ſuch parts, that one part may be three times as much above 20 as the other wants of 20: here $a=48$, $b=20$, $r=3$, $s=1$; for to ſay that the exceſs muſt be three times the defect, is no other than to ſay, that the exceſs muſt be to the defect as 3 to 1; the reſt is eaſy.

The GENERAL DEMONSTRATION.

1ſt, The greater part $\frac{ar-br-bs}{r-s}$, and the leſs part $\frac{br+bs-as}{r-s}$

being

being added together make $\frac{ar-as}{r-s}=a$: again, the excefs of the greater part above b, is $\frac{ar-br-bs}{r-s}-\frac{b}{1}=\frac{ar-br-bs-br+bs}{r-s}=\frac{ar-2br}{r-s}$, and the excefs of b above the lefs part, which is what the lefs part wants of b, is $\frac{b}{1}-\frac{br-bs+as}{r-s}=\frac{br-bs-br-bs+as}{r-s}=\frac{as-2bs}{r-s}$; therefore the excefs of one part above b is to what the other wants of b, as $\frac{ar-2br}{r-s}$ is to $\frac{as-2bs}{r-s}$, that is, as $ar-2br$ is to $as-2bs$, that is, as $r\times\overline{a-2b}$ is to $s\times\overline{a-2b}$, or as r to s. *Q. E. D.*

PROBLEM 10. (See art. 55.)

144. *There are two places whofe diftance from each other is* a, *and from whence two perfons fet out at the fame time with a defign to meet, one travelling at the rate of* p *miles in* q *hours, and the other at the rate of* r *miles in* s *hours: I demand how long and how far each travelled before they met.*

SOLUTION.

The number of hours travelled by each, x.

Miles travelled by the firft, $\frac{px}{q}$.

By the fecond, $\frac{rx}{s}$.

By them both, $\frac{px}{q}+\frac{rx}{s}$.

Equation, $\frac{px}{q}+\frac{rx}{s}=a$; therefore $px+\frac{qrx}{s}=aq$; therefore $psx+qrx=aqs$; therefore x (or the number of hours travelled by each) $=\frac{aqs}{ps+qr}$: now to find how many miles the firft travelled, fay, if in q hours he travelled p miles, how many will he travel in a number of hours equal to $\frac{aqs}{ps+qr}$? for a fourth number, I multiply the third number $\frac{aqs}{ps+qr}$ by the fecond p, and the product is $\frac{apqs}{ps+qr}$; this again I

divide

divide by the firſt number q, and the quotient is $\frac{aps}{ps+qr}$; for dividing the numerator divides the whole fraction: by the ſame way of reaſoning, the number of miles travelled by the other will be found to be $\frac{aqr}{ps+qr}$: therefore the whole number of miles travelled by them both is $\frac{aps+aqr}{ps+qr}$ $=a$, which demonſtrates the ſolution.

EXAMPLE.

Let the diſtance of the two places be 154 miles; let the firſt travel at the rate of 3 miles in 2 hours, and the ſecond after the rate of 5 miles in 4 hours; then we ſhall have $a=154$, $p=3$, $q=2$, $r=5$, $s=4$, $ps=12$, $qr=10$, $ps+qr=22$, $\frac{aqs}{ps+qr}=\frac{154\times2\times4}{22}=56$, $\frac{aps}{ps+qr}=\frac{154\times3\times4}{22}=84$, $\frac{aqr}{ps+qr}=\frac{154\times2\times5}{22}=70$: therefore each travelled 56 hours; the firſt travelled 84 miles, and the other 70.

SCHOLIUM.

If in the foregoing problem we change p into r and q into s, and *vice verſa*, the conſequence will be, that the firſt traveller will now travel at the ſame rate as the ſecond did before, and the ſecond at the ſame rate as the firſt did before; but the motion whereby theſe two travellers approach towards each other will ſtill be the ſame, and therefore the time this motion is performed in, that is, the time that each travelled, muſt ſtill be the ſame: let us then make the changes abovementioned, firſt in the expreſſion of the time, and ſee whether that expreſſion will ſtill continue the ſame; then let us make the ſame changes in the two expreſſions of the miles, and ſee whether by this means, theſe expreſſions will not be converted each into the other: firſt then, the expreſſion of the time, which is $\frac{aqs}{ps+qr}$, by changing p into r, and q into s, and *vice verſa*, becomes $\frac{asq}{rq+sp}$, which is the ſame as $\frac{aqs}{ps+qr}$; therefore the expreſſion of the time ſuffers no alteration by theſe changes: ſecondly, the number of miles travelled by the firſt was $\frac{aps}{ps+qr}$, which after the changes abovementioned, becomes $\frac{arq}{rq+sp}$, which is the ſame

as $\frac{aqr}{ps+qr}$, the miles travelled by the second; and therefore *è converso*, the expression $\frac{aqr}{ps+qr}$ will be changed into the expression $\frac{aps}{ps+qr}$; and thus will the case of the first traveller be changed into that of the second, and *vice versa*.

PROBLEM II. (See art. 46.)

145. *Suppose that* p *pounds of gold out of water weigh* q *pounds in water, and that* r *pounds of silver weigh* s *pounds in water; suppose also that a mass weighing* a *pounds, and consisting of both gold and silver, when weighed in water, weighs only* b *pounds: I demand the distinct quantities of gold and silver in the mass.*

SOLUTION.

The number of pounds of gold in the mass, x.
Of silver, $a-x$.

The weight of the former in water, $\frac{qx}{p}$.

Of the latter, $\frac{as-sx}{r}$.

Equation, $\frac{qx}{p}+\frac{as-sx}{r}=b$; therefore $qx+\frac{aps-psx}{r}=bp$; therefore $qrx+aps-psx=bpr$; therefore $qrx-psx=bpr-aps$; therefore x (or the number of pounds of gold in the mass) $=\frac{bpr-aps}{qr-ps}$; therefore $a-x$ (or the pounds of silver) $=\frac{a}{1}-\frac{bpr+aps}{qr-ps}=\frac{aqr-bpr}{qr-ps}$.

DEMONSTRATION.

First, the weight of the gold $\frac{bpr-aps}{qr-ps}$, and the weight of the silver $\frac{aqr-bpr}{qr-ps}$ being added together make $\frac{aqr-aps}{qr-ps}=a$.

2dly, We must find in the next place, how much each weighs in water, by saying, if p pounds of gold weigh q pounds in water, what will $\frac{bpr-aps}{qr-ps}$ weigh? and the answer will be $\frac{bqr-aqs}{qr-ps}$; in like manner, the

the weight of the ſilver in water will be found to be $\frac{aqs-bps}{qr-ps}$.

3dly, Add both theſe weights together, that is, $\frac{bqr-aqs}{qr-ps}$, and $\frac{aqs-bps}{qr-ps}$, and they will make $\frac{bqr-bps}{qr-ps}$, or b. *Q. E. D.*

PROBLEM 12.

146. *There are two pipes, whereof one will fill a ciſtern in the time* p, *and the other in the time* q: *In what time will they both fill it?*

SOLUTION.

Put x for the time ſought; then ſay, if in the time p the firſt pipe will diſcharge one ciſternful, how much will it diſcharge in the time x? and the anſwer will be $\frac{x}{p}$; and for the ſame reaſon, $\frac{x}{q}$ will repreſent the quantity diſcharged by the other pipe in the ſame time x; therefore $\frac{x}{p}+\frac{x}{q}$ will be the quantity diſcharged by them both; but by the problem, they ought both to diſcharge one ciſternful in this time; therefore we have this equation, $\frac{x}{p}+\frac{x}{q}=1$; therefore $x+\frac{px}{q}=p$; therefore $qx+px=pq$; therefore x (or the time ſought) $=\frac{pq}{p+q}$.

DEMONSTRATION.

1ſt, If in the time p the firſt pipe diſcharges one ciſternful, in the time $\frac{pq}{p+q}$ it will diſcharge the quantity $\frac{q}{p+q}$.

2dly, And for the ſame reaſon, in the ſame time $\frac{pq}{p+q}$, the other pipe will diſcharge the quantity $\frac{p}{p+q}$.

3dly, Therefore in the time $\frac{pq}{p+q}$, both pipes together will diſcharge the quantity $\frac{p+q}{p+q}=1$, that is, one ciſternful. *Q. E. D.*

PRO-

PROBLEM 13.

147. *One has a certain number* n, *of children whose ages are in arithmetical progression; the common difference of the progression is* d, *and the age of the eldest child is to that of the youngest as* r *to* s: *What are the ages of the eldest and youngest?*

SOLUTION.

Here I might put x for the age of the eldest child, and consequently $x-d$ for the age of the second, and so on; but the work will succeed better, if we put $x-d$ or $x-1d$ for the age of the first child, $x-2d$ for the age of the second, $x-3d$ for the age of the third, $x-4d$ for the age of the fourth, *&c*; for by this means, the number of every child, reckoning from the first inclusively, will also be the coefficient of d in the expression of his age; but according to this way of reckoning, the number of the last, or youngest child will be n; therefore his age will be $x-nd$; whence we have this proportion, $x-d$ is to $x-nd$ as r to s; and this equation, $rx-dnr=sx-ds$; therefore $rx-sx-dnr=-ds$; therefore $rx-sx=dnr-ds$; therefore $x=\frac{dnr-ds}{r-s}$; therefore $x-d$ (or the age of the eldest child) is $\frac{dnr-ds}{r-s}-\frac{d}{1}=\frac{dnr-dr}{r-s}$; and $x-nd$ (or the age of the youngest) is $\frac{dnr-ds}{r-s}-\frac{nd}{1}=\frac{dns-ds}{r-s}$.

DEMONSTRATION.

According to this computation, the age of the eldest is to that of the youngest as $\frac{dnr-dr}{r-s}$ is to $\frac{dns-ds}{r-s}$, that is, as $dnr-dr$ is to $dns-ds$, that is, as $nr-r$ is to $ns-s$, that is, as $r\times\overline{n-1}$ is to $s\times\overline{n-1}$, that is, as r to s. *Q. E. D.*

PROBLEM 14. (See art. 68.)

148. *What number is that, which being severally added to* a, b *and* c, *all given numbers, will make the three sums to be in continual geometrical proportion?* where the reader may, if he pleases, suppose a to be the greatest, and c the least of the three numbers given.

SOLUTION.

Put x for the number ſought; and then the proportion will ſtand thus,

$x+a$ is to $x+b$ as $x+b$ is to $x+c$.

Multiply extremes and means, and you will have this equation, $xx+ax+cx+ac=xx+2bx+bb$; throw away xx from both ſides, and you will have $ax+cx+ac=2bx+bb$; therefore $ax-2bx+cx+ac=bb$; therefore $ax-2bx+cx=bb-ac$; therefore x (or the number ſought ($=\frac{bb-ac}{a-2b+c}$.

DEMONSTRATION.

1ſt, If the number ſought $\frac{bb-ac}{a-2b+c}$ be added to the firſt given number a, the ſum will be $\frac{aa-2ab+bb}{a-2b+c}=\frac{\overline{a-b}\times\overline{a-b}}{a-2b+c}$, which I ſhall therefore call the firſt extreme.

2dly, If $\frac{bb-ac}{a-2b+c}$ be added to the ſecond number b, the ſum will amount to $\frac{bb-ac+ab-2bb+bc}{a-2b+c}=\frac{ab-bb-ac+bc}{a-2b+c}$; but for the better conceiving of this fraction, I divide the numerator into two parts, *viz.* $ab-bb$, and $-ac+bc$; then it is plain that the firſt part $ab-bb$ is the product of $\overline{a-b}\times b$, and the ſecond part $-ac+bc$ is the product of $\overline{a-b}\times -c$; therefore the whole numerator is the product of $\overline{a-b}\times\overline{b-c}$, and the middle term is $\frac{\overline{a-b}\times\overline{b-c}}{a-2b+c}$.

3dly, $\frac{bb-ac}{a-2b+c}$ added to the third given number c, makes $\frac{bb-2bc+cc}{a-2b+c}=\frac{\overline{b-c}\times\overline{b-c}}{a-2b+c}$, which therefore will be the laſt extreme.

4thly, We are now in the next place to enquire whether theſe three ſums be in continual geometrical proportion: now if we compare the two firſt terms together, we ſhall find that the firſt ſum is to the ſecond as $\frac{\overline{a-b}\times\overline{a-b}}{a-2b+c}$ is to $\frac{\overline{a-b}\times\overline{b-c}}{a-2b+c}$, that is, as $\overline{a-b}\times\overline{a-b}$ is to $\overline{a-b}\times\overline{b-c}$, that is, as $\overline{a-b}$ is to $\overline{b-c}$.

5thly,

5thly, If we compare the ſecond and third terms together, we ſhall find that the ſecond ſum is to the third as $\frac{\overline{a-b}\times\overline{b-c}}{a-2b+c}$ is to $\frac{\overline{b-c}\times\overline{b-c}}{a-2b+c}$, that is, as $\overline{a-b}\times\overline{b-c}$ is to $\overline{b-c}\times\overline{b-c}$, that is, as $\overline{a-b}$ is to $\overline{b-c}$.

6thly, Since then the firſt ſum is to the ſecond as $\overline{a-b}$ is to $\overline{b-c}$, and the ſecond ſum is to the third alſo as $\overline{a-b}$ is to $\overline{b-c}$, it follows, that the firſt ſum is to the ſecond as the ſecond is to the third, and conſequently, that the three ſums are in continual proportion. *Q. E. D.*

SCHOLIUM.

The laſt problem was, to find a number, which being ſeverally added to three given ones a, b and c, will make them continual proportionals: let us now change this problem into another, by changing the extreme numbers a and c one for the other; and then the queſtion will be, to find a number, which being ſeverally added to c, b and a, will make them continual proportionals: here it is plain, firſt, that the number ſought in this caſe, muſt be the ſame with the number ſought in the former; 2dly, that the middle proportional muſt alſo be the ſame here as there; and laſtly, that the extremes muſt be the ſame in both caſes, but interchanged one for the other: now if the expreſſions in the former caſe be juſt, the ſubſtitution of a and c one for the other ought to have the ſame effect in thoſe expreſſions as it had in the nature of the problem itſelf; let us try this, by actually changing a and c one for the other in all thoſe expreſſions, and let us ſee what will be the conſequence.

Firſt then, the number ſought in the former caſe was $\frac{bb-ac}{a-2b+c}$; put a for c and c for a, and then that expreſſion will be changed into this, $\frac{bb-ca}{c-2b+a}$; but $bb-ca$ is the ſame as $bb-ac$, and $c-2b+a$ is the ſame as $a-2b+c$; therefore the number ſought in this caſe is $\frac{bb-ac}{a-2b+c}$, the ſame as in the former.

2dly, The middle proportional in the former caſe was $\frac{\overline{a-b}\times\overline{b-c}}{a-2b+c}$; change a and c one for the other, and then the expreſſion will be $\frac{\overline{c-b}\times\overline{b-a}}{c-2b+a}$; but $\overline{c-b}\times\overline{b-a}$ is the ſame as $\overline{b-c}\times\overline{a-b}$ or $\overline{a-b}\times\overline{b-c}$, becauſe a change of the ſigns in both factors will have no more effect

effect upon the product, than if no such change had been made; and $c-2b+a$ is the same as $a-2b+c$, as was before observed; therefore the middle proportional in this case is $\frac{\overline{a-b}\times\overline{b-c}}{a-2b+c}$, which is the same as in the former.

3dly, The first extreme in the former case was $\frac{\overline{a-b}\times\overline{a-b}}{a-2b+c}$; put a and c for one another in this expression, and it will be changed into $\frac{\overline{c-b}\times\overline{c-b}}{c-2b+a}=\frac{\overline{b-c}\times\overline{b-c}}{a-2b+c}$; therefore the first extreme in this case is the same with the last in the former.

But I would not be mistaken in this *scholium*, or in any other of the like nature, as if I introduced them to confirm the solutions of the problems to which they are annexed; for those problems need no such confirmation: my chief design herein, is to shew the beauty and consistency of truth, the necessary connexion one truth has with another, and how much more clearly and distinctly this harmony is to be perceived in the mathematical sciences than in any other whatsoever; and yet after all, it is not impossible but that I may in a great measure lose my labour.

PROBLEM 15. (See art. 38.)

149. *What two numbers are those, whereof the greater is to the less as* p *to* q, *and the product of their multiplication is to their sum as* r *to* s?

SOLUTION.

Put x for the less number, and the greater will be found by saying, as q is to p, so is x the less number to $\frac{px}{q}$ the greater; whence their sum will be $\frac{px}{q}+\frac{x}{1}=\frac{px+qx}{q}$: on the other hand, if the greater number $\frac{px}{q}$ be multiplied into x, the product will be $\frac{pxx}{q}$; therefore the product of these two numbers will be to their sum as $\frac{pxx}{q}$ is to $\frac{px+qx}{q}$, that is, as px to $p+q$; but according to the problem, the product is to the sum as r to s; therefore px is to $p+q$ as r to s; whence we have this equation, $psx=pr+qr$; and x (the less num-

 ber

ber ſought) $=\frac{pr+qr}{ps}$; therefore $px=\frac{pr+qr}{s}$; for dividing the denominator multiplies the whole fraction; therefore $\frac{px}{q}$ (or the greater number) $=\frac{pr+qr}{qs}$.

DEMONSTRATION.

1ſt, The greater number is to the leſs as $\frac{pr+qr}{qs}$ is to $\frac{pr+qr}{ps}$; divide $pr+qr$ by itſelf, and the quotient will be 1; ſo that we may now ſay, that the greater number is to the leſs as $\frac{1}{qs}$ is to $\frac{1}{ps}$, that is, as $\frac{1}{q}$ is to $\frac{1}{p}$, that is as $\frac{p}{q}$ is to 1, that is, as p is to q.

2dly, The greater number $\frac{pr+qr}{qs}$ and the leſs $\frac{pr+qr}{ps}$ being added together make $\frac{pprs+pqrs+pqrs+qqrs}{pqss}=\frac{pprs+2pqrs+qqrs}{pqss}$; but $pp+2pq+qq=\overline{p+q}^2$; therefore the ſum of the two numbers ſought is $\frac{rs\times\overline{p+q}^2}{pqss}$.

3dly, The greater number $\frac{pr+qr}{qs}$ multiplied into the leſs $\frac{pr+qr}{ps}$ produces $\frac{rr\times\overline{p+q}^2}{pqss}$.

4thly, Therefore the product of the two numbers ſought is to their ſum as $\frac{rr\times\overline{p+q}^2}{pqss}$ is to $\frac{rs\times\overline{p+q}^2}{pqss}$, that is, as rr is to rs, or as r to s. *Q. E. D.*

PROBLEM 16. (See art. 63.)

150. *One draws a certain quantity of wine out of a full veſſel that held a certain number of gallons equal to* a; *and then recruiting the veſſel with water, takes a ſecond draught of as much wine and water together as before he did of wine; and ſo he goes on for four draughts one after another, always taking the ſame quantity at a draught, and then recruiting*

recruiting the veſſel with water; inſomuch that at laſt, the number of gallons of pure wine left in the veſſel was only equal to b: *I demand how much he took at every draught.*

SOLUTION.

For the number of gallons left in the veſſel after every draught, which was always the ſame, put x; then it is plain that at every draught, the whole quantity of liquor in the veſſel, whether it was pure wine, or wine and water mixt, muſt be diminiſhed in the proportion of a to x; therefore the wine muſt be diminiſhed in that proportion without being recruited; whence we have the following proportions; 1ſt, as a is to x, ſo is x, the quantity of pure wine left in the veſſel after the firſt draught, to $\frac{xx}{a}$, the wine left in the veſſel after the ſecond draught; 2dly, as a is to x, ſo is $\frac{xx}{a}$, the wine left after the ſecond draught, to $\frac{x^3}{a^2}$, the wine left after the third draught; laſtly, as a is to x, ſo is $\frac{x^3}{a^2}$, the wine left after the third draught, to $\frac{x^4}{a^3}$, the wine left after the fourth draught; but according to the problem, the quantity of wine left after the fourth draught was b; therefore $\frac{x^4}{a^3}=b$; therefore $x^4=a^3b,=a^4\times\frac{b}{a}$; make $\frac{b}{a}=s^4$, and then we ſhall have $x^4=a^4s^4$, and x (or the quantity of liquor left in the veſſel after every draught) $=as$; whence $a-x$ (or the quantity taken at every draught) will be $a-as=a\times\overline{1-s}$.

EXAMPLE.

Suppoſe the veſſel when full held 81 gallons, and that there were 16 gallons of pure wine left after the fourth draught; then we ſhall have $a=81$, $b=16$, $\frac{b}{a}$ or $s^4=\frac{16}{81}$, $ss=\frac{4}{9}$, $s=\frac{2}{3}$, $1-s=\frac{1}{3}$, $a\times\overline{1-s}=81\times\frac{1}{3}=27$; therefore 27 gallons were taken at every draught.

DEMONSTRATION.

Since the wine in the veſſel was diminiſhed at every draught in the proportion of a to x, that is, in the proportion of a to as, or of 1 to s, it follows, that as 1 is to s, ſo is x or as, the wine left after the firſt draught, to ass, the wine left after the ſecond draught; and for the ſame reaſon,

as 1 is to s ſo is as^2 to as^3, the wine left after the third draught; and laſtly, as 1 is to s ſo is as^3 to as^4, the wine left after the fourth draught; but $as^4 = a \times \frac{b}{a}$ by the ſuppoſition, $= b$; therefore the quantity of pure wine left in the veſſel after the fourth draught was b. *Q. E. D.*

N. B. As there are but few numbers which admit of an exact ſquare root, ſo there are fewer ſtill which admit of exact roots of higher kinds; but in the courſe of this work, I ſhall ſhew how to extract all roots with equal facility, and to as great a degree of exactneſs as is generally required, to wit, by a full table of logarithms.

PROBLEM 17.

151. *What two numbers are thoſe, the product of whoſe multiplication is* p, *and the quotient of the greater divided by the leſs is* q?

SOLUTION.

Put x for the greater number, and conſequently $\frac{p}{x}$ for the leſs; then will the quotient of the greater divided by the leſs be $\frac{xx}{p}$; but according to the problem, this quotient ought to be q; therefore $\frac{xx}{p} = q$; and $xx = pq$, and x (the greater number ſought) $= \sqrt{pq}$: again, ſince $xx = pq$, we have $\frac{pp}{xx} = \frac{pp}{pq} = \frac{p}{q}$; and $\frac{p}{x}$ (or the leſs number ſought) $= \sqrt{\frac{p}{q}}$; ſo that the greater of the two numbers ſought is $\sqrt{pq}$, and the leſs $\sqrt{\frac{p}{q}}$.

EXAMPLE.

Let the product of the two numbers ſought be 144, and the quotient of the greater divided by the leſs 16; then we ſhall have $p = 144$, $q = 16$, $pq = 144 \times 16$, $\sqrt{pq} = 12 \times 4 = 48$; $\frac{p}{q} = \frac{144}{16}$, $\sqrt{\frac{p}{q}} = \frac{12}{4} = 3$; therefore the numbers are 48 and 3.

DEMONSTRATION.

1ſt, pq multiplied into $\frac{p}{q}$ gives $\frac{ppq}{q} = pp$; therefore $\sqrt{pq}$ multiplied into $\sqrt{\frac{p}{q}}$ gives p.

2dly,

2dly, pq being divided by $\frac{p}{q}$ gives $\frac{pqq}{p}=qq$; therefore $\sqrt{pq}$ being divided by $\sqrt{\frac{p}{q}}$ gives q. *Q. E. D.*

PROBLEM 18.

152. *Let* x *and* y *be two unknown quantities, and* a, b, c, d, e, f *known quantities: It is required to find the values of* x *and* y *by the help of the two following equations, to wit,* $ax+by=c$, *and* $dx+ey=f$.

SOLUTION.

Equation 1ſt, $ax+by=c$.

Equation 2d, $dx+ey=f$. Multiply the firſt equation by d, and you will have $adx+bdy=cd$; multiply the ſecond equation by a, and you will have $adx+aey=af$; ſubtract the former equation thus produced from the latter, and you will have

Equ. 3d, $\overline{ae-bd}\times y=af-cd$; whence

Equ. 4th, $y=\frac{af-cd}{ae-bd}$.

Therefore $by=\frac{abf-bcd}{ae-bd}$; put this value inſtead of by in the firſt equation, and you will have $ax+\frac{abf-bcd}{ae-bd}=c$; multiply the whole into $\overline{ae-bd}$, and you will have $\overline{aae-abd}\times x+abf-bcd=ace-bcd$; throw away $-bcd$ from both ſides of the equation, and you will have $\overline{aae-abd}\times x+abf=ace$; divide the whole by a, and you will have $\overline{ae-bd}\times x+bf=ce$; therefore $\overline{ae-bd}\times x=ce-bf$; therefore $x=\frac{ce-bf}{ae-bd}$; and ſo the quantities x and y are at laſt found to be $x=\frac{ce-bf}{ae-bd}$, and $y=\frac{af-cd}{ae-bd}$.

I have here purſued the common method in finding out x; but after y was diſcovered, x might have been more eaſily and more readily determined thus: ſuppoſe the two fundamental equations had been theſe that follow; $bx+ay=c$, and $ex+dy=f$; the conſequence would have been, that x in this caſe would have come out the ſame as y in the laſt, and y here the ſame as x there; therefore changing a into b, and d into e, and *vice verſa*, will change y into x; now the expreſſion of y is $\frac{af-cd}{ae-bd}$; change

change in this expreſſion a into b, and d into e, and *vice versa*, and it will then ſtand thus, $\frac{bf-ce}{bd-ae}=\frac{ce-bf}{ae-bd}=x$.

By this and many other inſtances that might be produced, it might eaſily be made appear, that theſe ſpeculations are ſometimes no leſs uſeful in calculation, than they are entertaining to men of taſte and genius, though perhaps they may be found too ſubtil to be reliſhed by the raw and narrow minds of young beginners

For the better conceiving of the foregoing expreſſions, diſpoſe the quantities in the order following:

$$\begin{matrix} a & d \\ c & f \\ b & e \end{matrix}$$

Firſt put down a *and* d, *the two coefficients of* x, *one after the other in the ſame order as they ſtand in the two fundamental equations; under theſe, and in the ſame order, put down* c *and* f, *the two abſolute terms; and laſtly under theſe again, put down* b *and* e, *the two coefficients of* y; *this done, compare the firſt couple* a *and* d *with the ſecond* c *and* f, *multiplying the terms croſſwiſe, and* af—cd *will be the numerator of the value of* y; *again, compare the ſecond couple* c *and* f *with the third* b *and* e, *multiplying them in like manner, and* ce—bf *will be the numerator of* x; *laſtly, compare the firſt pair* a *and* d *with the laſt* b *and* e, *multiplying them as before, and* ae—bd *will be a common denominator to both the former numerators; ſo that the coefficients of* x *enter into the numerator of* y, *thoſe of* y *into the numerator of* x, *and both into the common denominator.*

EXAMPLE.

Let it be required to determine x and y by the help of the two following equations; $3x-4y=6$, and
$5x-6y=14$.
The terms being diſpoſed according to the foregoing method, will ſtand thus;

$$\begin{matrix} 3 & 5 \\ 6 & 14 \\ -4 & -6 \end{matrix}$$

Firſt then I compare the firſt pair 3 and 5 with the ſecond 6 and 14, thus; $3\times14=42$, $5\times6=30$, and the latter product ſubtracted from the former leaves 12 for the numerator of y; in the next place I compare the ſecond pair 6 and 14 with the third -4 and -6, thus; $6\times-6=-36$, $14\times-4=-56$, and the latter product ſubtracted from the former leaves $+20$, the numerator of x; laſtly, I compare the firſt couple 3 and 5 with the third -4 and -6, thus; $3\times-6=-18$, $5\times-4=-20$,

$=-20$, and the latter product subtracted from the former leaves $+2$ for a common denominator to both x and y; so that we have $x=\frac{20}{2}$ or 10, and $y=\frac{12}{2}$ or 6; which numbers 10 and 6 will answer the conditions proposed.

N. B. If any of the terms ax, by, dx, ey be wanting, the coefficients must be supplied with cyphers; as if ax be wanting, a must be supposed equal to o.

The GENERAL DEMONSTRATION.

1st, Since $x=\frac{ce-bf}{ae-bd}$, and $y=\frac{af-cd}{ae-bd}$, we shall have $ax=\frac{ace-abf}{ae-bd}$, and $by=\frac{abf-bcd}{ae-bd}$; therefore $ax+by=\frac{ace-bcd}{ae-bd}=c$.

2dly, $dx=\frac{cde-bdf}{ae-bd}$, and $ey=\frac{aef-cde}{ae-bd}$; therefore $dx+ey=\frac{aef-bdf}{ae-bd}=f$. *Q. E. D.*

PROBLEM 19.

153. *Two persons* A *and* B *were talking of their money: says* A *to* B, *give me* q *of your money, and I shall then have* r *times as much as you will have left; says* B *to* A, *give me* q *of your money, and I shall then have* s *times as much as you will have left: How much money had each?*

SOLUTION.

For *A*'s and *B*'s money put x and y respectively; and the fundamental equations will be

$$x+q=ry-qr, \text{ and}$$
$$y+q=sx-qs;$$

and these equations when reduced to the form for the last problem, will stand thus;

$$x-ry=-q-qr, \text{ and}$$
$$sx-y=+q+qs.$$

Dispose the coefficients and absolute terms as in the last problem, and they will stand thus;

1	s
$-q-qr$	$+q+qs$
$-r$	-1

Compare the first couple with the second, and you will have $1\times\overline{q+qs}=\overline{q+qs}$, and $s\times\overline{-q-qr}=\overline{-qs-qrs}$; subtract the latter product from the former, and you will have $q+2qs+qrs$ for the numerator of y; compare the second couple with the third, and you will have $\overline{-q-qr}\times-1=+q+qr$, and $\overline{q+qs}\times-r=-qr-qrs$; subtract

tract the latter product from the former, and you will have $q+2qr+qrs$ for the numerator of x; compare the first couple with the last, and you will have $1x-1=-1$, and $sx-r=-rs$; subtract the latter product from the former, and you will have $rs-1$ for a common denominator; therefore x (or *A*'s money) was $\frac{q+2qr+qrs}{rs-1}$; and y (or *B*'s money) was $\frac{q+2qs+qrs}{rs-1}$.

If q be taken equal to $rs-1$, it will destroy the denominator, and the solution will come out in whole numbers, thus; *A*'s money will be $1+2r+rs$, and *B*'s will be $1+2s+rs$; as for example, let $r=3$, and $s=7$; then you will have $rs-1=20$; put $q=20$, and the problem will stand thus; says *A* to *B*, give me 20 shillings of your money, and I shall then have three times as much as you will have left; says *B* to *A*, give me 20 shillings of your money, and I shall then have 7 times as much as you will have left: the answer to which is, that *A*'s money, or $1+2r+rs=28$ shillings; and *B*'s money, or $1+2s+rs=36$ shillings.

Note, That if *A* upon receiving q of *B*'s money had a sum equal to what *B* had left, in that case r must be made equal to 1.

The General Demonstration.

1st, If to *A*'s money, to wit, $\frac{q+2qr+qrs}{rs-1}$, be added q, the sum will be $\frac{2qr+2qrs}{rs-1}$; and if from *B*'s money, to wit, $\frac{q+2qs+qrs}{rs-1}$, be subtracted q, the remainder will be $\frac{2q+2qs}{rs-1}$, which if multiplied by r, will be equal to the former sum; and therefore the first condition of the problem is answered.

2dly, If to *B*'s money be added q, the sum will be found to be $\frac{2qs+2qrs}{rs-1}$; and if from *A*'s money be subtracted q, the remainder will be found to be $\frac{2q+2qr}{rs-1}$, which being multiplied by s, will be equal to the former sum; and therefore the second condition of the problem is answered. *Q. E. D.*

Problem 20. (See art. 81.)

154. *It is required to find two numbers* x *and* y *of such a nature, that if both be multiplied into* r, *the first product shall be a square, and the second*

ſecond the ſide or root of that ſquare; but if both be multiplied into s, *the firſt product ſhall be a cube, and the ſecond the root of that cube.*

S O L U T I O N.

The numbers x and y being multiplied both into r, produce rx and ry, whereof the former is to be the ſquare of the latter; whence we have this equation, $rx = r^2y^2$, and $x = ry^2$: again, x and y multiplied both into s, produce sx and sy, whereof the former is to be the cube of the latter; whence we have this equation, $sx = s^3y^3$ and $x = s^2y^3$; therefore $s^2y^3 = ry^2$, becauſe both are equal to x; divide both ſides by y^2, and you will have $s^2y = r$, and $y = \frac{r}{ss}$; but if $y = \frac{r}{ss}$, $y^2 = \frac{r^2}{s^4}$, and ry^2 or $x = \frac{r^3}{s^4}$; therefore the numbers are, $x = \frac{r^3}{s^4}$, and $y = \frac{r}{s^2}$.

D E M O N S T R A T I O N.

If the numbers $\frac{r^3}{s^4}$ and $\frac{r}{s^2}$ be multiplied both into r, their products will be $\frac{r^4}{s^4}$ and $\frac{r^2}{s^2}$, whereof the former is a ſquare, and the latter is the root of that ſquare; and if the ſame numbers $\frac{r^3}{s^4}$ and $\frac{r}{s^2}$ be multiplied both into s, the products will be $\frac{r^3}{s^3}$ and $\frac{r}{s}$, whereof the former is a cube, and the latter the root of that cube. *Q. E. D.*

P R O B L E M 21. (See art. 130.)

155. *What two numbers are thoſe, whoſe difference being multiplied into the difference of their ſquares will make* a, *and whoſe ſum being multiplied into the ſum of their ſquares will make* b?

S O L U T I O N.

For the two numbers ſought put x and y; then according to the firſt ſuppoſition, $\overline{x-y} \times \overline{x^2-y^2}$, or $\overline{x-y} \times \overline{x-y} \times \overline{x+y}$, or $\overline{xx-2xy+y^2} \times \overline{x+y} = a$; therefore

$$\text{Equ. 1ſt,} \quad x^2 - 2xy + y^2 = \frac{a}{x+y}.$$

Again, according to the ſecond ſuppoſition, $\overline{x+y} \times \overline{x^2+y^2} = b$; therefore

Equ. 2d, $x^2+y^2=\frac{b}{x+y}$.

From twice the second equation, subtract the first,

that is, from $2x^2 \quad * + 2y^2=\frac{2b}{x+y}$

subtract $x^2-2xy+y^2=\frac{a}{x+y}$

and there will remain $x^2+2xy+y^2=\frac{2b-a}{x+y}$,

that is, $\overline{x+y}^2=\frac{2b-a}{x+y}$; therefore $\overline{x+y}^3=2b-a$; make $2b-a=r^3$, that is, put r for the cube root of $2b-a$, and you will have

Equ. 3d, $x+y=r$.

Again, in the first equation we had $x^2-2xy+y^2=\frac{a}{x+y}=\frac{a}{r}$, that is, $\overline{x-y}^2=\frac{a}{r}$; make $\frac{a}{r}=ss$, that is, put s for the square root of $\frac{a}{r}$, and you will have

Equ. 4th, $x-y=s$.

Add the third and fourth equations together, and you will have $2x=r+s$, and $x=\frac{r+s}{2}$; subtract the fourth equation from the third, and you will have $2y=r-s$, and $y=\frac{r-s}{2}$; whence we have the following canon:

Make $2b-a=r^3$, *and* $\frac{a}{r}=s^2$, *and the numbers sought will be* $\frac{r+s}{2}$, *and* $\frac{r-s}{2}$.

DEMONSTRATION.

The difference of the numbers $\frac{r+s}{2}$ and $\frac{r-s}{2}$ is s, and the difference of their squares is rs, as is easily tryed; therefore the difference of the numbers multiplied into the difference of their squares is $rss=\frac{ra}{r}=a$: again, the sum of the numbers $\frac{r+s}{2}$ and $\frac{r-s}{2}$ is r, and the sum of

their

their ſquares is $\frac{r^2+s^2}{2}$; therefore the ſum of the numbers multiplied into the ſum of their ſquares is $\frac{r^3+rss}{2}$; but $r^3=2b-a$ by the canon, and $rss=a$ by the ſame; therefore the ſum of the numbers multiplied into the ſum of their ſquares is $\frac{2b-a+a}{2}=b$. *Q. E. D.*

PROBLEM 22.

156. *Out of a common pack of fiftytwo cards, let part be diſtributed into ſeveral diſtinct parcels or heaps in the manner following: upon the loweſt card of every heap let as many others be laid as are ſufficient to make up it's number twelve; as if four be the number of the loweſt card, let eight others be laid upon it; if five, let ſeven; if* a, *let twelve* —a, &c: *It is required, having given the number of heaps, which we ſhall call* n, *as alſo the number of cards ſtill remaining in the dealer's hand, which we ſhall call* r, *to find the ſum of the numbers of all the bottom cards put together.*

SOLUTION.

Let a, b, c, &c expreſs the number of the bottom card in the ſeveral heaps; then will $12-a$ expreſs the number of all the cards lying upon the bottom card of the firſt heap, that is, the number of all the cards of the firſt heap except the loweſt, will be $12-a$; therefore $13-a$ will be the number of all the cards in the firſt heap; for the ſame reaſon, $13-b$ will be the number of all the cards in the ſecond heap; and $13-c$ the number of all thoſe in the third, and ſo on; therefore the number of all the cards in all the heaps will be $13\times n-a-b-c$ &c: make $a+b+c$ &c (or the ſum of the numbers of all the bottom cards) $=x$, and then we ſhall have the number of all the cards drawn out into heaps $=13n-x$; but theſe, together with r, the number of cards undrawn out, make up the whole pack 52; therefore we have this equation, $13n-x+r=52$; therefore $x+52=13n+r$; therefore $x=13n-52+r$; but $52=13\times 4$; therefore $13n-52=13\times\overline{n-4}$; therefore $x=13\times\overline{n-4}+r$; in words thus: *From the number of heaps ſubtract four; multiply the reſt by thirteen; and this product added to the number of cards ſtill remaining in the dealer's hand, will give the ſum of the numbers of all the bottom cards put together:* as for example, let there be three heaps, and thirty cards remaining; now 4 ſubtracted from 3 leaves -1; this multiplied by 13 gives -13, and this product added

to 30, the number of cards remaining, gives 17 for the ſum of the numbers of all the bottom cards.

A more univerſal theorem is as follows:

Let n *be the number of heaps as before,* p *the number of cards in a pack; let as many cards be laid upon the loweſt of every heap as are ſufficient to make up it's number* q; *and laſtly, let* r *be the number of remaining cards as before; and the ſum of the numbers of all the bottom cards will be found to be* $\overline{q+1}\times n+r-p$.

PROBLEM 23.

157. *It is required to determine the values of three unknown quantities* x, y *and* z *by the help of three equations of the following form, to wit,* $px+qy+rz=s$, *where the quantities repreſented by* p, q, r, s; *are all ſuppoſed to be known.*

N. B. I know not whether I have not already been too tedious in my 18th problem, and therefore I ſhall not trouble my reader with the inveſtigation of the following theorem; I ſhall only give him a few hints, which, if at any time he attempts it himſelf, may be of ſome uſe to him: but firſt I ſhall give him the theorem itſelf, which is as follows:

SOLUTION.

Firſt put down the three coefficients of x *one after another as they ſtand in the three equations, which coefficients we will call* a, b, c; *under theſe, put down the three abſolute terms, ſuppoſe* d, e, f; *under theſe again write down the coefficients of* z, *ſuppoſe* g, h, k; *and under theſe write the coefficients of* y, *ſuppoſe* l, m, n: *from theſe four ſerieſes thus put down, muſt be derived two others, to wit,* α, β, γ, *and* $\delta, \varepsilon, \zeta$; *and from theſe two laſt a third, to wit,* η, θ, κ, *all which muſt be written down as in the following diagram* †; *and the terms of the three laſt ſerieſes muſt be had by a croſs multiplication thus: make* $ae-bd=\alpha$, $ah-bg=\beta$, $am-bl=\gamma$; *make alſo* $bf-ce=\delta$, $bk-ch=\varepsilon$, *and* $bn-cm=\zeta$; *laſtly, make* $\alpha\varepsilon-\delta\beta=\eta$, $\alpha\zeta-\delta\gamma=\theta$, *and* $\beta\zeta-\varepsilon\gamma=\kappa$, *and this laſt ſeries* η, θ, κ *will ſolve the problem: for* y *will be* $\frac{-\eta}{\kappa}$, *and* z *will be* $\frac{+\theta}{\kappa}$; *whence* x *will eaſily be had by any of the three fundamental equations, if inſtead of* y *and* z *be ſubſtituted their values here found.*

†

a	b	c
d	e	f
g	h	k
l	m	n
α		δ
β		ε
γ		ζ
	η	
	θ	
	κ	

Who-

Whoever attempts the inveſtigation of this theorem, had beſt do it by the method laid down in the 18th problem; for if $px+qy+rz=s$, we ſhall have $px+qy=s-rz$; where the quantity $s-rz$ muſt be looked upon as the abſolute term: this obſerved, if he makes uſe of the coefficients contained in the ſerieſes above deſcribed, and applies the method of the 18th problem to the two firſt equations, he will find the value of y, when contracted by our notation, to be $\frac{\alpha-\beta z}{\gamma}$; and if he applies the ſame method to the ſecond and third equations, he will find $y=\frac{\delta-\varepsilon z}{\zeta}$; therefore he will have this equation, $\frac{\alpha-\beta z}{\gamma}=\frac{\delta-\varepsilon z}{\zeta}$; which being reſolved and contracted, gives $z=\frac{\theta}{\kappa}$: and when he conſiders how the terms α, β, γ, &c were obtained, he will naturally fall into the method above deſcribed; and laſtly, when he has got this theorem for determining the value of z, he will as eaſily find the value of y, to wit, by putting the coefficients of y in the place of thoſe of z, and *vice verſa*.

N. B. Coming now to give ſome examples of the ſolution of general problems producing quadratic equations, I muſt advertiſe the reader once for all, that whenever I am to expreſs the ſquare root of any quantity, which properly ſpeaking, is no ſquare, to avoid the introducing of ſurd roots, I do it for the moſt part by the letter s, putting s^2 for the number whoſe ſquare root is to be ſignified by s: thus if I am to expreſs the ſquare root of $\frac{aa-4b}{4}$, becauſe the denominator 4 is a ſquare number, I exclude it out of the value of s^2, and ſo put s^2 for $aa-4b$, whence $\frac{aa-4b}{4}=\frac{s^2}{4}$, and it's root will be $\frac{s}{2}$; but if I was to expreſs the ſquare root of $\frac{4b-a^3}{12a}$, ſince here neither the numerator nor denominator are ſquare numbers, I might put ss for the whole fraction; but conſidering that $\frac{4b-a^3}{12a}$ is the product of $\frac{1}{4}$ multiplied into $\frac{4b-a^3}{3a}$, whereof the former factor is a ſquare, and the latter is no ſquare, I rather chooſe to put ss for the latter factor $\frac{4b-a^3}{3a}$, and ſo make $\frac{4b-a^3}{12a}=\frac{ss}{4}$.

Note alſo, That I ſhall reſolve all the following equations in the ordinary way, without having recourſe to the general theorem in art. 103.

PRO-

PROBLEM 24. (See art. 111.)

158. *What two numbers are those, whose sum is* a, *and the product of whose multiplication is* b?

SOLUTION.

The two numbers sought, x and $a-x$.

The product of their multiplication, $ax-xx=b$; whence changing the signs, $xx-ax=-b$, and compleating the square, $xx-ax+\frac{aa}{4}=\frac{aa}{4}-b=\frac{aa-4b}{4}=\frac{ss}{4}$; extract the square root of both sides, that is, of $x^2-ax+\frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x-\frac{a}{2}=\pm\frac{s}{2}$, and $x=\frac{a\pm s}{2}$; whence the following canon:

Make $aa-4b=ss$, *and the greater number will be* $\frac{a+s}{2}$, *and the less number* $\frac{a-s}{2}$.

The SYNTHETICAL DEMONSTRATION.

1st, $\frac{a+s}{2}$ added to $\frac{a-s}{2}$ gives $\frac{2a}{2}$ or a.

2dly, $\frac{a+s}{2}$ multiplied into $\frac{a-s}{2}$ gives $\frac{aa-ss}{4}=$ (by substituting $-aa+4b$ instead of $-ss$) $\frac{aa-aa+4b}{4}=\frac{4b}{4}=b$. *Q.E.D.*

An example to the foregoing canon.

What two numbers are those, whose sum is twentyfive, and the product of whose multiplication is 144? Here $a=25$, $b=144$, $aa-4b$ or $ss=49$, $s=7$, $\frac{a+s}{2}=16$, $\frac{a-s}{2}=9$; so the numbers are 9 and 16.

PROBLEM 25. (See art. 113.)

159. *What two numbers are those, whose sum is* a, *and the sum of their squares* b?

SOLUTION.

The two numbers ſought, x and $a-x$.
The ſquare of the former, xx.
The ſquare of the latter, $aa-2ax+xx$.
The ſum of their ſquares $aa-2ax+2xx=b$; therefore $2xx-2ax=b-aa$, and $xx-ax=\frac{b-aa}{2}$, and $xx-ax+\frac{aa}{4}=\frac{aa}{4}+\frac{b-aa}{2}=\frac{2b-aa}{4}=\frac{ss}{4}$; extract the ſquare roots, that is, the root of $xx-ax+\frac{aa}{4}$ on one ſide, and of $\frac{ss}{4}$ on the other, and you will have $x-\frac{a}{2}=\pm\frac{s}{2}$, and $x=\frac{a\pm s}{2}$; whence the following canon:

Make $2b-aa=ss$, *and you will have* $\frac{a+s}{2}$ *for the greater number, and* $\frac{a-s}{2}$ *for the leſs.*

DEMONSTRATION.

1ſt, $\frac{a+s}{2}$ added to $\frac{a-s}{2}$ gives a.

2dly, The ſquare of $\frac{a+s}{2}$ is $\frac{aa+2as+ss}{4}$; the ſquare of $\frac{a-s}{2}$ is $\frac{aa-2as+ss}{4}$; and therefore the ſum of their ſquares is $\frac{2aa+2ss}{4}=\frac{aa+ss}{2}=$ (by the canon) $\frac{aa+2b-aa}{2}=b$. *Q. E. D.*

An example to the foregoing canon.

What two numbers are thoſe, whoſe ſum is 28, and the ſum of their ſquares 400? Here $a=28$, $b=400$, $2b-aa$ or $ss=16$, $s=4$, $\frac{a+s}{2}=16$, $\frac{a-s}{2}=12$; therefore the numbers are 12 and 16.

PROBLEM 26. (See art. 114.)

160. *What two numbers are thoſe, whoſe ſum is* a, *and the ſum of their cubes* b?

SOLU-

SOLUTION.

The two numbers ſought, x and $a-x$.
The cube of the former, x^3.
The cube of the latter, $a^3-3a^2x+3ax^2-x^3$.
The ſum of their cubes, $a^3-3a^2x+3ax^2=b$; therefore $3ax^2-3a^2x=b-a^3$; divide by $3a$, and you will have $xx-ax=\frac{b-a^3}{3a}$, and $xx-ax+\frac{aa}{4}=\frac{aa}{4}+\frac{b-a^3}{3a}=\frac{4b-a^3}{12a}=\frac{1}{4}\times\frac{4b-a^3}{3a}=\frac{ss}{4}$; extract the ſquare root of both ſides, that is, of $xx-ax+\frac{aa}{4}$ on one ſide, and of $\frac{ss}{4}$ on the other, and you will have $x-\frac{a}{2}=\pm\frac{s}{2}$ and $x=\frac{a\pm s}{2}$; whence the following canon:

Make $\frac{4b-a^3}{3a}=ss$, *and you will have* $\frac{a+s}{2}$ *for the greater number, and* $\frac{a-s}{2}$ *for the leſs.*

DEMONSTRATION.

1ſt, $\frac{a+s}{2}$ added to $\frac{a-s}{2}$ gives a.

2dly, The cube of $\frac{a+s}{2}$ is $\frac{a^3+3a^2s+3as^2+s^3}{8}$, and the cube of $\frac{a-s}{2}$ is $\frac{a^3-3a^2s+3as^2-s^3}{8}$; therefore the ſum of their cubes is $\frac{2a^3+6as^2}{8}=\frac{a^3+3ass}{4}=\frac{a^3+4b-a^3}{4}$ by the canon, $=b$. *Q. E. D.*

An example to the foregoing canon.

What two numbers are thoſe, whoſe ſum is 7, and the ſum of their cubes 133? Here $a=7$, $b=133$, $\frac{4b-a^3}{3a}$ or $ss=9$, $s=3$, $\frac{a+s}{2}=5$, $\frac{a-s}{2}=2$; ſo the numbers are 5 and 2.

PROBLEM 27.

161. *It is required to find two numbers whoſe difference is* d, *and which dividing a given number as* a, *will have two quotients whoſe difference is* b.

SOLU-

SOLUTION.

The two numbers ſought, x and $x+d$.

The two quotients, $\frac{a}{x}$ and $\frac{a}{x+d}$.

Their difference, $\frac{a}{x}-\frac{a}{x+d}=\frac{ad}{xx+dx}=b$; therefore $bxx+bdx=ad$, and $xx+dx=\frac{ad}{b}$; therefore $xx+dx+\frac{dd}{4}=\frac{ad}{b}+\frac{dd}{4}=\frac{1}{4}\times\overline{\frac{4ad}{b}+dd}=\frac{ss}{4}$; extract the ſquare root of $xx+dx+\frac{dd}{4}$ on one ſide, and of $\frac{ss}{4}$ on the other, and you will have $x+\frac{d}{2}=\pm\frac{s}{2}$, whence $x=\frac{s-d}{2}$ or $\frac{-s-d}{2}$; ſet aſide the negative root, and you will have x (the leſs diviſor) $=\frac{s-d}{2}$, and $x+d$ (the greater) $=\frac{s-d}{2}+\frac{d}{1}=\frac{s+d}{2}$; and we ſhall have the following canon:

Make $\frac{4ad}{b}+dd=ss$, *and you will have* $\frac{s+d}{2}$ *for the greater diviſor, and* $\frac{s-d}{2}$ *for the leſs.*

N. B. That $\frac{s-d}{2}$ is an affirmative quantity, is evident from hence, that $ss=\frac{4ad}{b}+dd$; therefore ss is greater than dd, and s greater than d; therefore $\frac{s-d}{2}$ is affirmative.

The demonſtration of the canon.

1ſt, If the leſs diviſor $\frac{s-d}{2}$ be ſubtracted from the greater $\frac{s+d}{2}$, the remainder will be d; therefore the difference of the diviſors is d.

2dly, If the dividend a be ſeverally divided by the two diviſors $\frac{s-d}{2}$ and $\frac{s+d}{2}$, the two quotients will be $\frac{2a}{s-d}$ and $\frac{2a}{s+d}$ reſpectively, whereof the former will be the greater, as having a leſs denominator; therefore

fore the difference of the quotients is $\frac{2a}{s-d}-\frac{2a}{s+d}=\frac{2as+2ad-2as+2ad}{ss-dd}$ $=\frac{4ad}{ss-dd}=\frac{4ad}{\frac{4ad}{b}}$ by the canon, $=b$. Q. E. D.

An example to the foregoing canon.

Let it be required to find two divisors whose difference is 1, and which dividing a given number as 144, will have two quotients whose difference is 2. Here $a=144$, $b=2$, $d=1$, $\frac{4ad}{b}+dd$ or $ss=289$, $s=17$, $\frac{s+d}{2}=9$, $\frac{s-d}{2}=8$; therefore the divisors are 8 and 9, and the quotients 18 and 16.

SCHOLIUM.

If in this last problem we had put x for the greater quantity, and $x-d$ for the less, the equation would have been $\frac{a}{x-d}-\frac{a}{x}=b$, or $\frac{ad}{xx-dx}$ $=b$, which is different from the former; and therefore it could not be expected, that in that equation, the two roots should be the numbers sought, but rather the two different values of x, the lesser of them.

PROBLEM 28. (See art. 118.)

162. *What number is that, which being added to it's square root, will make* a?

SOLUTION.

Put xx for the number sought, and you will have this equation, $xx+1x=a$; therefore $xx+1x+\frac{1}{4}=a+\frac{1}{4}=\frac{4a+1}{4}=\frac{ss}{4}$; therefore $x+\frac{1}{2}=\pm\frac{s}{2}$; therefore $x=\frac{s-1}{2}$ or $\frac{-s-1}{2}$: If x be made $=\frac{s-1}{2}$, you will have $xx=\frac{ss-2s+1}{4}$; if x be made equal to $\frac{-s-1}{2}$, you will have $xx=\frac{ss+2s+1}{4}$; whence the following canon:

Make $4a+1=ss$, *and the number sought will be* $\frac{ss-2s+1}{4}$ *or* $\frac{ss+2s+1}{4}$,

$\frac{ss+2s+1}{4}$, *according as the square root to be added is taken affirmatively or negatively.*

DEMONSTRATION.

Case 1st, If to the number $\frac{ss-2s+1}{4}$ be added it's affirmative square root $\frac{s-1}{2}$, or $\frac{2s-2}{4}$, the sum will be $\frac{ss-1}{4}=a$, by the canon.

Case 2d, If to the number $\frac{ss+2s+1}{4}$ be added it's negative square root $\frac{-s-1}{2}$ or $\frac{-2s-2}{4}$, the sum will again be $\frac{ss-1}{4}=a$, as before. *Q. E. D.*

PROBLEM 29. (See art. 129.)

163. *It is required to find three numbers in continual proportion, whose sum is* a, *and the sum of their squares* ab.

Therefore b (or $\frac{ab}{a}$) is the quotient of the sum of the squares divided by the sum of the numbers; which way of notation will be of some small advantage in the computation that follows.

SOLUTION.

For the three numbers sought put x, y and z; then since these numbers are in continual proportion, that is, since x is to y as y is to z, we have $yy=xz$; again, since by the supposition, $x+y+z=a$, we have $a-y=x+z$, and squaring both sides, $a^2-2ay+yy=xx+2xz+zz$; subtract y^2 from one side, and it's equal xz from the other, and you will have $aa-2ay=xx+xz+z^2=x^2+y^2+z^2=ab$; since then $aa-2ay=ab$, divide by a and you will have $a-2y=b$, and y (the middle term) $=\frac{a-b}{2}$; subtract this from a, the sum of all the three numbers sought, and you will have $\frac{a+b}{2}$ for the sum of the extremes; let us call this sum $2l$, and let us also call $\frac{a-b}{2}$ or the middle term, m; then as x is one of the extremes, $2l-x$ will be the other; and the product of the extremes will be $2lx-xx=m^2$; therefore $xx-2lx=-m^2$, and $xx-2lx+l^2=l^2-m^2=n^2$, and $x-l=\pm n$, and $x=l\pm n$; whence may be derived the following canon:

Make $\frac{a+b}{2}=2l$, $\frac{a-b}{2}=m$, *and* $l^2-m^2=n^2$, *and the three numbers sought will be* $\overline{l+n}$, m *and* $\overline{l-n}$.

DEMONSTRATION.

1st, Since by the supposition $n^2=l^2-m^2$, we have $m^2=l^2-n^2=\overline{l+n}\times\overline{l-n}$; therefore the three numbers $\overline{l+n}$, m and $\overline{l-n}$ are in continual proportion, by the latter end of the 15th article, because the square of the middle term is equal to the product of the extremes; which answers the first condition of the problem.

2dly, If the three numbers $\overline{l+n}$, m, and $\overline{l-n}$ be added together, their sum will be $2l+m=\frac{a+b}{2}+\frac{a-b}{2}$ by the canon, $=a$; which answers the second condition: in like manner, if m or $\frac{a-b}{2}$ be subtracted from $2l$ or $\frac{a+b}{2}$, you will have $2l-m=b$.

3dly, The square of $\overline{l+n}$ is $l^2+2ln+n^2$, the square of m is m^2, and the square of $\overline{l-n}$ is $l^2-2ln+n^2$; add these three squares together, and their sum will be $2l^2+m^2+2n^2=2l^2+m^2+2l^2-2m^2$ by the canon, $=4l^2-m^2=\overline{2l+m}\times\overline{2l-m}=ab$, by the second step; therefore all the conditions of the problem are satisfied. *Q. E. D.*

An example to the foregoing canon.

Let the sum of the three numbers be 19, and let the sum of their squares be 133; then we shall have $a=19$, $b=\frac{133}{19}=7$, $\frac{a+b}{2}$ or $2l=13$, $l=\frac{13}{2}$, $\frac{a-b}{2}$ or $m=6$, l^2-m^2 or $n^2=\frac{25}{4}$, $n=\frac{5}{2}$, $l+n=9$, $l-n=4$; therefore the numbers are 9, 6 and 4, or 4, 6 and 9.

PROBLEM 30.

164. *It is required to find four numbers in continual proportion, and such, that the sum of the extremes may be* a, *and that of the middle terms* b.

Note, For what is meant by 4 numbers being in continual proportion, see art. 128.

SOLU-

SOLUTION.

For the two middle terms put x and y; then the extremes will be found, one by saying, as y is to x so is x to $\frac{xx}{y}$, and the other by saying, as x is to y so is y to $\frac{yy}{x}$; so the extremes are $\frac{xx}{y}$ and $\frac{yy}{x}$, and their sum $\frac{x^2}{y}+\frac{yy}{x}=\frac{x^3+y^3}{xy}$; therefore the fundamental equations are, 1st, $x+y=b$, or $x=b-y$; and 2dly, $\frac{x^3+y^3}{xy}=a$, or $x^3+y^3=axy$; instead of x in this last equation use $b-y$, it's value in the first, and you will have $x^3=b^3-3bby+3byy-y^3$, and $x^3+y^3=b^3-3bby+3byy$; you will also have axy, or $ay\times\overline{b-y}=aby-ayy$; therefore the equation will now be $3byy-3bby+b^3=aby-ayy$, transpose b^3, as also $aby-ayy$, and then the equation will stand thus, $\overline{ayy+3byy-aby-3bby}=-b^3$, or thus, $\overline{yy-by}\times\overline{a+3b}=-b^3$; therefore $yy-by=\frac{-b^3}{a+3b}$, and $yy-by+\frac{bb}{4}=\frac{bb}{4}-\frac{b^3}{a+3b}=\frac{abb-b^3}{4a+12b}=\frac{bb}{4}\times\frac{a-b}{a+3b}=\frac{bb}{4}\times ss=\frac{bbss}{4}$; extract the square root of $yy-by+\frac{bb}{4}$ on one side, and of $\frac{bbss}{4}$ on the other, and you will have $y-\frac{b}{2}=\pm\frac{bs}{2}$, and $y=\frac{b}{2}\pm\frac{bs}{2}=\frac{b}{2}\times\overline{1\pm s}$; and since the equation will be the same, which soever of the two middle terms y is made to stand for, the greater or the less, we have the following canon:

Make $\frac{a-b}{a+3b}=ss$, *and you will have* $\frac{b}{2}\times\overline{1+s}$ *for the greater of the two middle terms, and* $\frac{b}{2}\times\overline{1-s}$ *for the less.*

The middle terms then being thus obtained by the help of the foregoing canon, the extremes will easily be had from the nature of continual proportionality as above, to wit, by saying, *As* $\frac{b}{2}\times\overline{1-s}$ *is to* $\frac{b}{2}\times\overline{1+s}$, *or as* $1-s$ *is to* $1+s$, *so is* $\frac{b}{2}\times\overline{1+s}$ *to* $\frac{b}{2}\times\frac{\overline{1+s}^2}{1-s}$; *therefore* $\frac{b}{2}\times\frac{\overline{1+s}^2}{1-s}$ *is*

is the greater extreme: and as $\frac{b}{2}\times\overline{1+s}$ *is to* $\frac{b}{2}\times\overline{1-s}$, *or as* $1+s$ *is to* $1-s$, *ſo is* $\frac{b}{2}\times\overline{1-s}$ *to* $\frac{b}{2}\times\frac{\overline{1-s}^2}{1+s}$; *therefore* $\frac{b}{2}\times\frac{\overline{1-s}^2}{1+s}$ *is the leſſer extreme; and the four numbers, putting the greater extreme firſt, will ſtand thus;* $\frac{b}{2}\times\frac{\overline{1+s}^2}{1-s}$, $\frac{b}{2}\times\overline{1+s}$, $\frac{b}{2}\times\overline{1-s}$ *and* $\frac{b}{2}\times\frac{\overline{1-s}^2}{1+s}$.

DEMONSTRATION.

That theſe four numbers muſt be continual proportionals is plain; for the extremes were found upon that ſuppoſition: and that the two middle terms when added together muſt make b is as plain; for $\frac{b}{2}\times\overline{1+s}$ $+\frac{b}{2}\times\overline{1-s}=\frac{b}{2}\times 2=b$; therefore the main buſineſs of this demonſtration will be to prove that the ſum of the extremes is a; we are therefore to enquire whether $\frac{b}{2}\times\frac{\overline{1+s}^2}{1-s}+\frac{b}{2}\times\frac{\overline{1-s}^2}{1+s}=a$, or dividing by $\frac{b}{2}$, whether $\frac{\overline{1+s}^2}{1-s}+\frac{\overline{1-s}^2}{1+s}=\frac{2a}{b}$: now $\frac{\overline{1+s}^2}{1-s}$ and $\frac{\overline{1-s}^2}{1+s}$ being added together make $\frac{\overline{1+s}^3+\overline{1-s}^3}{1-ss}$; but $\overline{1+s}^3=1+3s+3ss+s^3$, and $\overline{1-s}^3$ $=1-3s+3ss-s^3$; therefore $\frac{\overline{1+s}^3+\overline{1-s}^3}{1-ss}=\frac{2+6ss}{1-ss}$; therefore the enquiry is now reduced to this, whether $\frac{2+6ss}{1-ss}=\frac{2a}{b}$: now to determine this, both the numerator $2+6ss$, and the denominator $1-ss$, muſt be examined apart thus; $ss=\frac{a-b}{a+3b}$ by the canon; therefore $6ss=\frac{6a-6b}{a+3b}$; therefore $2+6ss$ or the numerator $=\frac{2}{1}+\frac{6a-6b}{a+3b}$ $=\frac{8a}{a+3b}$: again, $ss=\frac{a-b}{a+3b}$; therefore $-ss=\frac{-a+b}{a+3b}$; therefore

$1-ss,$

$1-ss$, or the denominator, equals $\frac{1}{1} - \frac{a+b}{a+3b} = \frac{4b}{a+3b}$; therefore $\frac{2+6ss}{1-ss}$ is equal to a fraction whose numerator is $\frac{8a}{a+3b}$, and whose denominator is $\frac{4b}{a+3b}$; but such a fraction is equal to $\frac{8a}{4b}$ or $\frac{2a}{b}$; therefore $\frac{2+6ss}{1-ss}$, or the sum of the extremes after being divided by $\frac{b}{2}$, equals $\frac{2a}{b}$; therefore before any such division was made, the sum of the extremes was a. *Q. E. D.*

An example to the foregoing canon.

Let the sum of the extremes be 27, that of the middle terms 18; and you will have $a=27$, $b=18$, $\frac{a-b}{a+3b}$ or $ss=\frac{9}{81}=\frac{1}{9}$, $s=\frac{1}{3}$, $1+s=\frac{4}{3}$, $1-s=\frac{2}{3}$, $\frac{b}{2}\times\overline{1+s}=12$, $\frac{b}{2}\times\overline{1-s}=6$; and so the two middle terms are 12 and 6; whence the extremes are 24 and 3; and the 4 numbers in order are 24, 12, 6 and 3, or 3, 6, 12 and 24.

PROBLEM. 31.

165. *What two numbers are those, whose sum added to the sum of their squares is* a, *and whose difference added to the difference of their squares is* b?

SOLUTION.

Put x and y for the two numbers sought, and the fundamental equations will be 1st, $x+y+x^2+y^2=a$; 2dly, $x-y+x^2-y^2=b$; which equations when reduced to order will stand thus;

Equ. 1st, $xx+x+yy+y=a$.

Equ. 2d, $xx+x-yy-y=b$.

Add these two last equations together, and you will have $2xx+2x=a+b$; whence $xx+1x=\frac{a+b}{2}$, and $xx+1x+\frac{1}{4}=\frac{a+b}{2}+\frac{1}{4}=\frac{2a+2b+1}{4}$ $=\frac{rr}{4}$; extract the root of $xx+1x+\frac{1}{4}$ on one side, and of $\frac{rr}{4}$ on the other, and you will have $x+\frac{1}{2}=\frac{r}{2}$, and $x=\frac{r-1}{2}$: again, subtract the second equation from the first, and you will have $2y^2+2y=a-b$;

$-b$; and $y^2+y=\frac{a-b}{2}$, and $y^2+1y+\frac{1}{4}=\frac{2a-2b+1}{4}=\frac{ss}{4}$; whence $y+\frac{1}{2}=\frac{s}{2}$, and $y=\frac{s-1}{2}$; whence the following canon:

Make $2a+2b+1=rr$, *and* $2a-2b+1=ss$, *and you will have the greater number equal to* $\frac{r-1}{2}$, *and the less number* $=\frac{s-1}{2}$.

DEMONSTRATION.

The sum of $\frac{r-1}{2}$ and $\frac{s-1}{2}$ is $\frac{r+s-2}{2}$ or $\frac{2r+2s-4}{4}$.

The square of $\frac{r-1}{2}$ is $\frac{r^2-2r+1}{4}$.

The square of $\frac{s-1}{2}$ is $\frac{s^2-2s+1}{4}$.

Therefore the sum of their squares is $\frac{r^2+s^2-2r-2s+2}{4}$; add to this the sum of the numbers above found, to wit, $\frac{2r+2s-4}{4}$, and you will have the sum of the numbers added to the sum of their squares equal to $\frac{r^2+ss-2}{4}$; but $r^2+s^2=4a+2$ by the canon; therefore $rr+ss-2=4a$, and $\frac{r^2+s^2-2}{4}$, or the sum of the numbers added to the sum of their squares, equals a: again, the difference of $\frac{r-1}{2}$ and $\frac{s-1}{2}$ is $\frac{r-s}{2}$ or $\frac{2r-2s}{4}$; and the difference of their squares is $\frac{r^2-s^2+2s-2r}{4}$; therefore the difference of the numbers added to the difference of their squares is $\frac{r^2-s^2}{4}=\frac{4b}{4}$ by the canon, $=b$. *Q. E. D.*

An example to the foregoing canon.

Let the sum of the numbers added to the sum of their squares be 26, and their difference added to the difference of their squares 14; and we shall have $a=26$, $b=14$, $2a+2b+1$ or $rr=81$, $r=9$, $\frac{r-1}{2}=4$, $2a-2b+1$ or $ss=25$, $s=5$, $\frac{s-1}{2}=2$; and so the numbers sought will be 4 and 2.

PRO-

PROBLEM 32.

166. *What two numbers are those, the sum of whose squares is* a, *and the product of their multiplication* b?

SOLUTION.

For the two numbers sought put x and $\frac{b}{x}$, and the sum of their squares will be $x^2 + \frac{b^2}{x^2} = a$; therefore $x^4 + b^2 = ax^2$; therefore $x^4 - ax^2 = -bb$, and $x^4 - axx + \frac{aa}{4} = \frac{aa}{4} - bb = \frac{aa - 4bb}{4} = \frac{ss}{4}$; extract the square root of $x^4 - ax^2 + \frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x^2 - \frac{a}{2} = \pm\frac{s}{2}$, and $x^2 = \frac{a \pm s}{2}$; and since this equation will be the same, whichsoever of the unknown quantities x is made to stand for, you will have the following canon:

Make $aa - 4bb = ss$, *and you will have the square of the greater number equal to* $\frac{a + s}{2}$, *and the square of the less equal to* $\frac{a - s}{2}$.

DEMONSTRATION.

If the square of the greater number, which is $\frac{a + s}{2}$, be added to the square of the less number, which is $\frac{a - s}{2}$, the sum of their squares will be $\frac{2a}{2}$ or a: again, if the square of the greater number which is $\frac{a + s}{2}$ be multiplied into the square of the less number which is $\frac{a - s}{2}$, the product of these two squares will be $\frac{aa - ss}{4} = \frac{aa - aa + 4bb}{4}$ by the canon, $= \frac{4bb}{4} = bb$; but if the square of the greater number multiplied into the square of the less gives bb, then the greater number multiplied into the less will give b. *Q. E. D.*

An example to the foregoing canon.

Let the ſum of the ſquares of the two numbers ſought be 400, and the product of their multiplication 192; then you will have $a = 400$, $b = 192$, $a^2 - 4b^2$ or $s^2 = 12544$, $s = 112$, $\frac{a+s}{2}$ or the ſquare of the greater number $= 256$, $\frac{a-s}{2}$ or the ſquare of the leſs number $= 144$; therefore the greater number is 16, and the leſs 12.

PROBLEM 33.

167. *Let a ſtone be dropt into an empty pit or well: It is required, having given the time from the firſt dropping of the ſtone to the hearing of the ſound from the bottom, to aſſign the depth of the well.*

SOLUTION.

The ſolution of this problem is founded upon three philoſophical principles, all which muſt here be taken for granted: The firſt is, that the ſpaces through which a heavy body falls from reſt in different times, are as the ſquares of thoſe times: thus the ſpace deſcribed in two ſeconds of time, is to the ſpace deſcribed in three ſeconds, not as 2 to 3, but as 4 to 9. The ſecond is, that ſetting aſide the reſiſtence of the air, which is not here conſidered, all bodies both great and ſmall, heavy and light, are accelerated alike, and deſcend through the ſame ſpace in the ſame time. The third is, that all ſounds, at leaſt, all ſuch as are ſtrong enough to affect our ſenſes, are propagated through the air with the ſame velocity, and conſequently deſcribe ſpaces proportionable to the time of their motion. The firſt of theſe principles was demonſtrated long ſince by it's firſt diſcoverer *Galileo*, and the two laſt by the celebrated Sir *Iſaac Newton*. Theſe things then being ſuppoſed, let x be put for the depth of the well, let a repreſent the ſpace through which a heavy body falls from reſt in one ſecond of time, b the ſpace through which a ſound paſſes in the ſame time, and let c be the time given in ſeconds from the firſt deſcent of the ſtone to the hearing of the ſound; then to find how long the ſtone muſt be in deſcending to the bottom of the well, ſay, as the ſpace a is to the ſpace x, to wit, the whole depth of the well, ſo is 1, the ſquare of the time wherein the former ſpace is deſcribed, to wit, the ſquare of one ſecond, to $\frac{x}{a}$, the ſquare of the time wherein the lat-

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ter ſpace is deſcribed; therefore $\frac{\surd x}{\surd a}$ will repreſent the time the ſtone takes in deſcending through the whole depth of the well: again, to find the time the ſound takes in aſcending through the ſame ſpace, ſay, as the ſpace b is to the ſpace x, ſo is one ſecond, the time wherein the former ſpace is deſcribed, to $\frac{x}{b}$ the time wherein the latter ſpace is deſcribed: but theſe times together, that is, the time of the ſtone's deſcent, and the time of the ſound's aſcent, make up the whole time c; therefore we have this equation, $\frac{x}{b} + \frac{\surd x}{\surd a} = c$; therefore $x + \frac{b\surd x}{\surd a} = bc$. This is a ſort of a quadratic equation, ſince x is the ſquare of $\surd x$; or at leaſt it is in the form of a quadratic equation, and muſt be reſolved accordingly. Here then the coefficient of the ſecond term is $\frac{-b}{\surd a}$, whoſe half is $\frac{b}{2\surd a}$; the ſquare of this is $\frac{bb}{4a}$, which being added to both ſides of the equation, gives $x + \frac{b\surd x}{\surd a} + \frac{bb}{4a} = \frac{bb}{4a} + bc = \frac{bb + 4abc}{4a}$: make $bb + 4abc = ss$, and you will have $x + \frac{b\surd x}{\surd a} + \frac{bb}{4a} = \frac{ss}{4a}$; whence extracting the root of both ſides, you will have $\surd x + \frac{b}{2\surd a} = \pm \frac{s}{2\surd a}$; therefore $\surd x = \frac{-b \pm s}{2\surd a}$, that is, $\surd x = \frac{-b - s}{2\surd a}$, or $\frac{-b + s}{2\surd a}$, whereof the former root is negative, and the latter affirmative; but the nature of this problem will not admit that $\surd x$ be ſuppoſed negative; for we muſt be conſiſtent with ourſelves, and if we ſuppoſe $\surd x$ to be negative in any one part of this ſolution, it muſt be ſo in all the reſt; whence it will follow, that $\frac{\surd x}{\surd a}$, the time of the deſcent of the ſtone through the depth of the well is negative, which is abſurd; it is for this reaſon that I reject the negative root $\frac{-s - b}{2\surd a}$, and retain the affirmative one $\frac{s - b}{2\surd a}$, making $\surd x = \frac{s - b}{2\surd a}$, and conſequently x the depth of the well $= \frac{\overline{s - b}^2}{4a}$.

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If any one asks, how I came to know that even the accepted root $\frac{s-b}{2\sqrt{a}}$ was affirmative; this was easily discovered by observing, that ss being equal to $bb+4abc$, will be greater than bb; therefore s will be greater than b, and consequently $s-b$ will be an affirmative quantity, as must also $\frac{s-b}{2\sqrt{a}}$: this might also have been discovered by turning over all the parts of the equation to one side, and so making $\frac{x}{b}+\sqrt{\frac{x}{a}}-c=0$; see art. 109; but to proceed: since $x=\frac{\overline{s-b}^{2}}{4a}$, we shall have $\sqrt{x}=\frac{s-b}{2\sqrt{a}}$, and $\frac{\sqrt{x}}{\sqrt{a}}$, or the time of the descent of the stone to the bottom of the well, equal to $\frac{s-b}{2a}$; we shall also have $\frac{x}{b}$, which was the time of the sound's ascent through the same space, equal to $\frac{\overline{s-b}^{2}}{4ab}$; whence is derived the following canon:

Make $bb+4abc$ *equal to* ss, *and you will have the depth of the well equal to* $\frac{\overline{s-b}^{2}}{4a}$, *the time of the stone's descent equal to* $\frac{s-b}{2a}$, *and the time of the sound's ascent equal to* $\frac{\overline{s-b}^{2}}{4ab}$.

DEMONSTRATION.

If the depth of the well be rightly assigned, the time of the stone's descent, and the time of the sound's ascent added together, ought to make up the whole time c, from the dropping of the stone to the hearing of the sound: now the time of the stone's descent was $\frac{s-b}{2a}$, or $\frac{2bs-2bb}{4ab}$, multiplying both the numerator and denominator by $2b$; and the time of the sound's ascent was $\frac{\overline{s-b}^{2}}{4ab}$, or $\frac{s^2-2bs+bb}{4ab}$; add these two times together, to wit, $\frac{s^2-2bs+bb}{4ab}$, and $\frac{2bs-2bb}{4ab}$, and the sum will be $\frac{s^2-b^2}{4ab}=\frac{4abc}{4ab}$ by the canon, $=c$. Q. E. D.

SCHO-

SCHOLIUM.

Monsieur *Huygens*, justly celebrated for having been the author of so many curious and useful discoveries, both to the learned world in particular, and to mankind in general, having observed that the length of a pendulum that swings seconds is 3 feet 8 lines and a half, *Paris* measure; and having also by an uncommon sagacity found out this curious theorem, that the time of the descent of a heavy body through half the length of any pendulum, is to the time of one single oscillation of that pendulum either forwards or backwards, as the diameter of a circle is to the circumference; from both these together, and from *Galileo*'s theorem above mentioned, has demonstrated, that the space through which a heavy body falls from rest in one second of time, is $15\frac{1}{12}$ *Paris* feet, or more accurately, 15.0957 *Paris* feet, that is, 16.1222 English feet, increasing the number of *Paris* feet in the proportion of 1000 to 1068. This discovery, I say, is owing to the sagacity of that excellent Mathematician Monsieur *Huygens*; nor do we owe less to the care and diligence of our learned countreyman Mr. *Derham*, who in his experiments and observations upon the motion of sounds, from a far greater distance than any before him had ever observed them, found it to be at the rate of about 1142 feet in a second: therefore 16.1222 feet is the space which in the resolution of the last problem we called a, and 1142 feet is what we called b; therefore $4a = 64.4888$, and $4ab = 73646$, and $bb = 1304164$. Having then got b^2 and $4ab$ ready calculated to our hand, which in all cases will be the same, the solution of the foregoing problem will become much easier: as for example, suppose the time from the beginning of the fall to the hearing of the sound was 10 seconds; then we shall have $c = 10$, $4abc = 736460$, $bb + 4abc$ or $ss = 2040624$, $s = 1428.5$, $s - b = 286.5$, $\overline{s-b}^2 = 82082.25$; this therefore divided by $4a$ or 64.49, gives 1273 feet for the depth of the well: moreover the time of the stone's descent was $\frac{s-b}{2a}$ by the canon; but $s-b$ is found already to be 286.5; and this divided by $2a$ or 32.24, gives 8.89 seconds for the time of the stone's descent: lastly, the time of the sound's ascent was $\frac{\overline{s-b}^2}{4ab}$ according to the canon; and $\frac{\overline{s-b}^2}{4a}$ was found before to be 1273; divide this by b or 1142, and you will have $\frac{\overline{s-b}^2}{4ab}$ or

or the time of the found's afcent equal to 1.11 of a fecond: add this to the former time, to wit, 8.89 feconds, and the whole time will amount to 10 feconds, as it ought.

If any one be defirous to fee, how the fpace through which a heavy body defcends from reft in one fecond of time is deduced from *Huygens*'s theorem above mentioned, it is done as follows: let l be the length of a pendulum that fwings feconds; and then according to that theorem, the time wherein a heavy body defcends from reft through a fpace equal to $\frac{1}{2}l$, will be to the time of one ofcillation of the pendulum whofe length is l, as the diameter of a circle is to the circumference; but the time of one ofcillation of the pendulum l is one fecond by the fuppofition; or if we put s for the unknown fpace through which a heavy body falls from reft in one fecond of time, then the time of one ofcillation of the pendulum l will be the fame with the time of defcent through the fpace s, and the proportion will now ftand thus; the time of defcent through $\frac{1}{2}l$ will be to the time of defcent through s, as the diameter of a circle is to the circumference; and therefore the fquare of the former time will be to the fquare of the latter, as the fquare of the diameter is to the fquare of the circumference; but the fquare of the time of defcent through $\frac{1}{2}l$ is to the fquare of the time of defcent through s, as $\frac{1}{2}l$ is to s, according to what has been delivered above; therefore $\frac{1}{2}l$ is to s, as the fquare of the diameter of a circle is to the fquare of the circumference; but the diameter of a circle is to the circumference as 113 to 355, (fee fchol. 1. to art. 179;) and the fquare of the former is to the fquare of the latter as 12769 is to 126025; moreover l the length of a pendulum that fwings feconds is found by experience to be 3 feet $8\frac{1}{2}$ lines, *Paris* meafure, as above, that is, (by the introduction art. 23) 3.059028 feet; whence $\frac{1}{2}l = 1.529514$ of a foot; fay then, as 12769 is to 126025, fo is 1.529514 to 15.0957; and you will have $s = 15.0957$ French feet, or 16.1222 Englifh feet.

Hence may be deduced by way of corollary, that as the fquare of the diameter of a circle is to the fquare of the circumference, fo is half the length of any pendulum to the fpace through which a heavy body will fall from reft during the time of one ofcillation of that pendulum.

THE
SOLUTION OF TWO PROBLEMS

By Mr. ABRAHAM DE MOIVRE.

The curious and elegant ſolutions of the two following problems about proportionals were communicated by that moſt excellent Mathematician Mr. *de Moivre*, and his conſent has been obtained for the printing of them. They are here inſerted altogether ſuch as Mr. *de Moivre* ſent them, exactly in his own words.

PROBLEM I.

The ſum of four continual proportionals being given, as alſo the ſum of their ſquares, to find the proportionals.

SOLUTION.

Let the four proportionals be x^3, xxy, xyy, y^3 : let their ſum be $=a$, and the ſum of their ſquares $=b$: we have therefore the two following equations; $x^3+xxy+xyy+y^3=a$, and

$x^6+x^4yy+xxy^4+y^6=b$.

The firſt equation is reduced to this;

$\overline{x+y}\times\overline{xx+yy}=a$: the ſecond to

$\overline{xx+yy}\times\overline{x^4+y^4}=b$.

The firſt equation reduced, determines me to ſuppoſe

$x+y=z$, and

$xx+yy=v$:

Therefore the firſt equation is changed into

$zv=a$.

Now for the ſecond, I ſay, ſince $x+y=z$, then ſquaring all,

$xx+2xy+yy=zz$;

but $xx+yy$ has been ſuppoſed $=v$;

therefore $2xy+v=zz$,

and $2xy=zz-v$,

and $4xxyy=z^4-2vzz+vv$:

Again, ſince $xx+yy=v$, then ſquaring all,

$x^4+2xxyy+y^4=vv$:

but the ſecond equation reduced is

$\overline{xx+yy}\times\overline{x^4+y^4}=b$, or

$v\times\overline{x^4+y^4}=b$;

therefore $x^4+y^4=\dfrac{b}{v}$;

there-

therefore inſtead of the equation we had a little before, which was $x^4 + 2xxyy + y^4 = vv$, we may write

$$\frac{b}{v} + 2xxyy = vv, \text{ or}$$

$$2xxyy = vv - \frac{b}{v}, \text{ or}$$

$$4xxyy = 2vv - \frac{2b}{v}:$$ but we had found before, that $4xxyy = z^4 - 2vzz + vv$; therefore we have now the

equation $$2vv - \frac{2b}{v} = z^4 - 2vzz + vv, \text{ or}$$

$$vv - \frac{2b}{v} = z^4 - 2vzz:$$

But by the tranſformation of the firſt equation we have had $zv = a$.

therefore $$v = \frac{a}{z};$$

and this value of v being ſubſtituted, we ſhall have

$$\frac{aa}{zz} - \frac{2bz}{a} = z^4 - 2az, \text{ or}$$

$$z^6 \begin{matrix} + \frac{2b}{a} z^3 \\ - 2az^3 \end{matrix} = aa,$$ which is properly a quadratic.

Now z being found in that equation, then v will be found: when z and v are known, then x and y will be known; for 'tis plain that

$$x = \frac{1}{2} z + \frac{1}{2} \sqrt{2v - zz}, \text{ and}$$

$$y = \frac{1}{2} z - \frac{1}{2} \sqrt{2v - zz};$$ as will appear, if we conſider

that by ſuppoſition $x + y = z$, and

$$xx + yy = v:$$

Square the firſt of theſe two equations, and we ſhall have

$xx + 2xy + yy = zz$; but the ſecond equation is

$xx \quad + yy = v$; from which taking their difference,

$2xy = zz - v$; we have

$$\therefore xx - 2xy + yy = v - zz + v = 2v - zz;$$

hence $$x - y = \sqrt{2v - zz};$$

but $$x + y = z;$$

therefore $$2x = z + \sqrt{2v - zz},$$

and

and $$x=\frac{1}{2}z+\frac{1}{2}\sqrt{2v-zz}.$$

Again, it appears that $2y=z-\sqrt{2v-zz}$;

therefore $$y=\frac{1}{2}z-\frac{1}{2}\sqrt{2v-zz}.$$

Therefore the four proportionals are known.

PROBLEM 2.

The ſum of five proportionals, and the ſum of their ſquares being given, to find the proportionals.

SOLUTION.

Let the ſum of the proportionals be a, and the ſum of their ſquares b; then we have theſe two equations;

$$x^4+x^3y+xxyy+xy^3+y^4=a, \text{ and}$$
$$x^8+x^6yy+x^4y^4+xxy^6+y^8=b.$$

The firſt may be reduced to $\frac{x^5-y^5}{x-y}=a$;

the ſecond to $$\frac{x^{10}-y^{10}}{xx-yy}=b.$$

Divide the ſecond of theſe equations by the firſt; then we ſhall have

$\frac{x^5+y^5}{x+y}=\frac{b}{a}$; which equation being expanded, gives

$$x^4-x^3y+xxyy-xy^3+y^4=\frac{b}{a}.$$

Write under this, the firſt original equation, which was

$$x^4+x^3y+xxyy+xy^3+y^4=a.$$

Let theſe two equations be added together; then we ſhall have

$$2x^4+2xxyy+2y^4=a+\frac{b}{a}, \text{ or}$$

$$x^4+xxyy+y^4=\tfrac{1}{2}a+\frac{\frac{1}{2}b}{a}=s,$$

(ſuppoſing for abbreviation, that $\frac{1}{2}a+\frac{\frac{1}{2}b}{a}=s$:)

But if, inſtead of adding thoſe equations, we now ſubtract the uppermoſt from the lowermoſt, we ſhall have

$$2x^3y+2xy^3=a-\frac{b}{a}, \text{ or}$$

$$x^3y+xy^3=\tfrac{1}{2}a-\frac{\frac{1}{2}b}{a}, \text{ or}$$

$xy \times \overline{xx+yy} = \frac{1}{2}a - \frac{\frac{1}{2}b}{a}$; which known quantities to shorten, we may call d: we have therefore the equation

$$xy \times \overline{xx+yy} = d.$$

Here I am naturally determined to suppose $xy = z$, and $xx+yy = v$; therefore instead of the last equation,

I have $zv = d$.

My business now is to express the other equation, $x^4 + xxyy + y^4 = s$ by means of z and v: but since $xx+yy = v$, then squaring all, we have

$x^4 + 2xxyy + y^4 = vv$; from which,

subtracting $x^4 + xxyy + y^4 = s$, which we had before; then there remains $xxyy = vv - s$:

But $xy = z$; therefore $xxyy = zz$, and therefore

$zz = vv - s$. But we had before, $zv = d$:

Let z be exterminated; then we shall have

$$\frac{dd}{vv} = vv - s, \text{ or } dd = v^4 - svv.$$

Now v may easily be found in this equation; v being known, z will be known; both z and v being known, x and y will be known; for

$$x = \frac{1}{2}\sqrt{v+2z} + \frac{1}{2}\sqrt{v-2z}$$

$y = \frac{1}{2}\sqrt{v+2z} - \frac{1}{2}\sqrt{v-2z}$, as will appear if we consider that by supposition $xy = z$, and

$xx+yy = v$:

For doubling the first of these two equations, we shall have $2xy = 2z$; and this being added to the second, we have $xx + 2xy + yy = v + 2z$, therefore $x+y = \sqrt{v+2z}$.

Again, if we subtract the equation $2xy = 2z$ from the other $xx+yy=v$, we shall have $xx - 2xy + yy = v - 2z$; wherefore $x - y = \sqrt{v-2z}$.

Now from the two equations,

$x+y = \sqrt{v+2z}$,

and $x-y = \sqrt{v-2z}$, it is plain that

$$x = \frac{1}{2}\sqrt{v+2z} + \frac{1}{2}\sqrt{v-2z}$$

and $y = \frac{1}{2}\sqrt{v+2z} - \frac{1}{2}\sqrt{v-2z}$.

N. B.

N. B. It perhaps may be proper for the sake of the young Analyst, to subjoyn an example to each of the foregoing solutions communicated by Mr. *De Moivre.*

Example to the solution of the first problem.

Since $z^6 = \frac{2aa - 2b}{a} z^3 + aa$, if for $\frac{aa - b}{a}$ we put d, then will $z^6 = 2dz^3 + aa$, and $z^3 = d + \sqrt{dd + aa}$: let $a = 30$, $b = 340$; then will $aa = 900$, $aa - b = 560$, $\frac{aa - b}{a}$ or $d = \frac{56}{3}$, $aa + dd = 900 + \frac{3136}{9} = \frac{11236}{9}$, $\sqrt{aa + dd} = \frac{106}{3}$, $d + \sqrt{aa + dd}$ or $z^3 = 54 = 27 \times 2 = 27c^3$ (putting c for the cube root of 2,) $z = 3c$, $\frac{a}{z}$ or $v = \frac{30}{3c} = \frac{10}{c} = \frac{10cc}{c^3} = 5cc$, $2v - zz = cc$, $\sqrt{2v - zz} = c$, $\frac{z + \sqrt{2v - zz}}{2}$ or $x = 2c$, $\frac{z - \sqrt{2v - zz}}{2}$ or $y = c$;

whence the proportionals	x^3,	xxy,	xyy,	y^3,
are	$8c^3$,	$4c^3$,	$2c^3$,	$1c^3$,
that is,	16,	8,	4,	2.

Example to the solution of the second problem.

Let $a = 62$, $b = 1364$; then $aa = 3844$, $\frac{aa + b}{2a}$ or $s = 42$, $\frac{aa - b}{2a}$ or $d = 20$, $v^4 = 42vv + 400$, $vv = 21 + \sqrt{841} = 50 = 25 \times 2 = 25cc$ (putting c for the square root of 2,) $v = 5c$, $\frac{d}{v}$ or $z = \frac{20}{5c} = \frac{4}{c} = \frac{4c}{cc} = 2c$, $\sqrt{v + 2z} = \sqrt{9c} = 3\sqrt{c}$, $\sqrt{v - 2z} = \sqrt{c}$, $\frac{1}{2}\sqrt{v + 2z} + \frac{1}{2}\sqrt{v - 2z}$ or $x = 2\sqrt{c}$, $\frac{1}{2}\sqrt{v + 2z} - \frac{1}{2}\sqrt{v - 2z}$ or $y = \sqrt{c}$, $xx = 4c$, $xy = 2c$, $yy = c$;

and the proportionals	x^4,	x^3y,	$xxyy$,	xy^3,	y^4,
will be	$16cc$,	$8cc$,	$4cc$,	$2cc$,	$1cc$,
or	32,	16,	8,	4,	2.

THE ELEMENTS of ALGEBRA

BOOK V.

In what cases a problem may admit of many answers.

168. IT has already been observed, that if in any problem the number of independent conditions be equal to the number of unknown quantities, such a problem will admit but of one solution; or if it admits of more, they will however be so determined as to leave no room for arbitrary positions: but if the conditions be fewer in number than are the unknown quantities, those that are wanting may then be supplied by the Analyst himself at pleasure; and as there is infinite choice, it is no wonder if in such a case a problem admits of an infinite number of answers, especially where fractions are taken into that number; but if the problem relates to whole numbers only, then the number of answers will sometimes be finite and sometimes infinite, as the nature of the problem will bear. This will be sufficiently illustrated by the two following examples.

EXAMPLE I.

Let it be required to find two numbers whose sum is equal to ten times their difference.

Here putting x and y for the two numbers sought, it is plain that in this case we have but one condition, and consequently but one equation, to wit, $x+y=10x-10y$, which equation being reduced, gives $x=\frac{11y}{9}$; and this is all the problem requires. Here then it is plain that the Analyst is entirely at liberty to substitute whatever whole number, mixt number, or proper fraction he pleases for y, provided he does but make $x=\frac{11y}{9}$; and the two quantities x and y will solve the problem.

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As for inſtance, let $\frac{1}{2}$ be put for y; then will x or $\frac{11y}{9}$ be $\frac{11}{18}$, and thoſe two fractions $\frac{11}{18}$ and $\frac{1}{2}$ or $\frac{9}{18}$ will ſolve the problem; for their difference is $\frac{1}{9}$, and their ſum $\frac{10}{9}$. But if it be intended that x and y ſhall both be whole numbers, then ſuch a whole number muſt be ſubſtituted for y as will admit of 9 for a diviſor without a remainder: but of ſuch whole numbers there is infinite choice, as 9, 18, 27, 36 &c; therefore this queſtion is capable of an infinite number of anſwers, both in whole numbers and fractions.

EXAMPLE 2.

Let it now be required to find two numbers x *and* y, *the product of whoſe multiplication is equal to ten times their difference.*

Here the equation will be $yx = 10x - 10y$, which being reduced, gives $x = \frac{10y}{10-y}$. Here it is plain that y muſt be leſs than 10; for if y was equal to 10, the fraction $\frac{10y}{10-y}$ would be infinite, as will be ſhewn in another place; and if y be greater than 10, then $10-y$, and conſequently $\frac{10y}{10-y}$ will be a negative quantity, whereas the problem may be ſuppoſed to relate to affirmative quantities only: however, as there is infinite choice of fractions between 0 and 10, and as any of theſe may be ſubſtituted for y, the problem will ſtill be capable of an infinite number of ſolutions, if fractions may be admitted; but if it be required that x and y be both whole numbers, then there cannot be above nine ſuch numbers that can be put for y; nor perhaps all theſe neither, as remains in the next place to be ſhewn. Now to find what whole number being put for y will bring out x a whole number alſo, I reduce the quantity $\frac{10y}{10-y}$ to a more ſimple one, by dividing $10y$ by $10-y$, or rather by $-y+10$, beginning with $-y$ thus: $10y$ divided by $-y$ quotes -10, which I put down in the quotient; then multiplying the diviſor $-y+10$ by -10 the quotient, I find the product to be $+10y - 100$, which being ſubtracted from $10y$ the dividend, leaves 100 for a remainder; but not intending to carry on the diviſion any further, I repreſent the reſt of the quotient by the fraction $\frac{100}{10-y}$; ſo $x = \frac{100}{10-y} - 10$; therefore that x may be a whole number, it is neceſſary that $\frac{100}{10-y}$ be a whole number; but this will be impoſſible, unleſs $10-y$ be

be ſome one of the diviſors of 100, I mean ſuch a number as will divide 100 without remainder: I enquire therefore in the next place, how many ſuch diviſors 100 will admit of that are under 10; for ſo long as y is any thing, $10-y$ muſt be leſs than 10; and I find four ſuch diviſors, to wit, 1, 2, 4 and 5; therefore if $10-y$ be put equal to any of theſe, x or $\frac{100}{10-y}-10$ muſt come out a whole number; and it muſt alſo come out affirmative; for ſo long as $10-y$ is greater than nothing and leſs than 10, $\frac{100}{10-y}$ will always be greater than $\frac{100}{10}$, that is, than 10, and conſequently $\frac{100}{10-y}-10$ or x will be affirmative. Let us then ſuppoſe firſt, $10-y=1$, and we ſhall have $y=9$, and $\frac{10y}{10-y}$ or $x=90$. 2dly, if $10-y=2$, we ſhall have $y=8$, and $x=40$. 3dly, if $10-y=4$, we ſhall have $y=6$, and $x=15$. Laſtly, if $10-y=5$, we ſhall have $y=5$, and $x=10$: therefore this queſtion admits of 4 ſolutions in whole numbers, to wit, 90 and 9, 40 and 8, 15 and 6, and 10 and 5; all which equally anſwer the condition of the problem, as will appear upon tryal.

Thus having given the reader a taſte of what he is to expect under this head, I ſhall now make ſome preparation for the ſolution of ſuch problems as uſually fall under this denomination.

DEFINITIONS.

169. *If any one quantity can be meaſured or divided by another of the ſame kind without remainder, the former is called a multiple of the latter;* as 12 is a multiple of 4, and 6 is a multiple of 6; for in this ſenſe, every quantity is a multiple of itſelf, becauſe every quantity meaſures itſelf: *but if any one quantity can be ſo meaſured or divided by two or more others, it is then ſaid to be a common multiple of thoſe others*; as 60 is a common multiple of the numbers 2, 3, 4, 5, 6.

COROLLARY.

Hence, if any two unequal quantities a *and* b *be both multiples of a third, their ſum* a + b, *and their difference* a — b *or* b — a, *will alſo be multiples of the ſame; and if* a *and* b *be common multiples of any number of quantities, their ſum and difference will alſo be common multiples of the ſame;* for whatever quantities will meaſure both *a* and *b*, the ſame will meaſure both their ſum and difference, (ſee art. 20, ax. 2d;) therefore

fore *e converso*, whatever quantity or quantities *a* and *b* are multiples or common multiples of, the same will their sum and difference be multiples or common multiples of.

N. B. The following lemmas contain some curious properties, first of whole numbers, and then of fractions, worthy to be treated of as well for their own sakes, as for the use they have in the resolution of some of the following problems.

LEMMA I.

A THEOREM.

170. *Let* a *and* b *be any two quantities whatever whose least common multiple is* c*: I say then, that this least common multiple* c *shall measure any other common multiple* d *of the same quantities.*

If this be denied, it must then be allowed that after *c* has measured *d* as often as is possible, there must still remain a quantity as *e* less than *c*: be it so, and then *c* will measure *d*—*e*. Since then *a* and *b* both measure *c*, and *c* measures *d*—*e*, *a* and *b* will both measure *d*—*e*; therefore *d*—*e* is a common multiple of *a* and *b*; but *d* is a common multiple of the same by the supposition; therefore both *d* and *d*—*e* are common multiples of *a* and *b*; therefore their difference *e* is a common multiple of the same, by art. 169; but *e* is less than *c* by the supposition; therefore if *c* does not measure *d* without a remainder, it will be possible for two quantities *a* and *b* to have a common multiple less than their least; but this is impossible; therefore *c* does measure *d* without a remainder. *Q. E. D.*

COROLLARY.

If the demonstration here given had been applied to three or more quantities, it would equally have proved that their least common multiple will measure any other common multiple whatever of the same quantities.

SCHOLIUM.

This way of demonstrating the truth of a proposition by running it's contrary into an absurdity, or into an impossibility, or into a contradiction to something already established and demonstrated, was with a great deal of subtilty practiced by the Ancients whenever a more direct proof could not so easily be come at; and it is for that reason, that this sort of demonstrations is sometimes less embarrassed than those of the direct kind, where less subtilty is required: and certainly they are as firm, and as conclusive as any others whatever, as depending upon the same axioms and principles; and if they be not altogether so convincing, it must be

because

because the machinery and contrivance of them does not so immediately appear, especially to young Mathematicians, who cannot as yet be supposed to be acquainted with all the different modes of thinking and reasoning upon quantity: therefore whenever I can, from a doubtful proposition, by just reasoning, deduce another that is apparently false, it may serve for an infallible mark to shew, that the first position from whence the deduction was made, was not true; for truth can lead to nothing but truth.

LEMMA 2.

A PROBLEM.

171. *It is required, having given any two unequal numbers* a *and* b, *to find their least common multiple.*

SOLUTION.

As ab, the product of a and b multiplied together, is a common multiple of both, it may be divided by their least common multiple without a remainder, by art. 170: imagine it then to be so divided, and let the quotient be c; then *e converso*, if the product ab be divided by c, the quotient $\frac{ab}{c}$ will be the least common multiple of a and b: whence I infer, that $\frac{a}{c}$ and $\frac{b}{c}$ will be whole numbers; for else, how should $\frac{ab}{c}$ be any common multiple of a and b, much less their least common multiple? but if $\frac{a}{c}$ and $\frac{b}{c}$ be whole numbers, then c must be a common measure of a and b; and I say in the next place, that it must be their greatest common measure; for should any number greater than c, suppose d, measure them both; since $\frac{a}{d}$ and $\frac{b}{d}$ would be whole numbers, $\frac{ab}{d}$ would be a common multiple of a and b less than $\frac{ab}{c}$, because d is greater than c, and so a and b would have a common multiple less than their least, which is absurd: therefore, *If* a *and* b, *the two numbers first proposed, be multiplied together, and their product divided by their greatest common measure, the quotient will be their least common multiple.* Q. E. I.

Thus if the numbers 12 and 15 be proposed, whose greatest common measure is 3; the product of their multiplication is 180, and this divided by 3 gives 60 for their least common multiple.

COROL-

COROLLARY 1.

Hence if the numbers given be prime to each other, that is, are such as will admit of no common measure greater than unity, the product of their multiplication will be their least common multiple: for in this case, $\frac{ab}{c}=\frac{ab}{1}=ab$.

COROLLARY 2.

If the fraction $\frac{\mathrm{a}}{\mathrm{b}}$ *be reduced to it's least terms* $\frac{\mathrm{d}}{\mathrm{e}}$; *I say then that* ae *or* bd *(for they are equal) will be the least common multiple of* a *and* b: for in this case, if c be the greatest common measure of a and b, we shall have $d=\frac{a}{c}$, and $e=\frac{b}{c}$, and therefore ae or bd will be $\frac{ab}{c}$.

LEMMA 3.

A PROBLEM.

172. *It is required, having given three or more numbers, to find their least common multiple.*

Let the numbers given be a, b, c and d, and the solution will be as follows: *First by the foregoing article find the least common multiple of* a *and* b, *and call it* e; *then find the least common multiple of* e *and* c, *and call it* f; *then find the least common multiple of* f *and* d, *and call it* g: *I say then that the number* g *last found will be the least common multiple of the numbers* a, b, c *and* d. For first, that g will be a common multiple of those numbers I thus demonstrate: a and b must measure e, their least common multiple; and e will measure f, the least common multiple of e and c; and f will measure g, the least common multiple of f and d; therefore both a and b will measure g: again, c will measure f, the least common multiple of e and c; and f measures g as before; therefore c measures g; but d measures g *ex hypothesi*; therefore all the given numbers a, b, c and d measure g; therefore g is a common multiple of them all. I say in the next place that g is their least common multiple: for if it be possible, let h be a less, that is, let all the numbers a, b, c and d measure a number h that shall be less than g; and since h is a common multiple of a and b *ex hypothesi*, and e is their least common multiple, e must measure h by art. 170; but c measures h *ex hypothesi*; therefore since both e and c measure h, their least common multiple f must also measure h; but d measures h *ex hypothesi*; therefore since both f and d measure h, their least common multiple g must also measure h, that is, a

 greater

greater quantity must measure a less, which is absurd; therefore g is the least common multiple of the numbers given. *Q. E. I.*

Of this solution take the following example. Let it be required to find the least common multiple of the numbers 8, 10, 12 and 14: now according to the last article, the least common multiple of 8 and 10 is 40; and the least common multiple of 40 and 12 is 120; and the least common multiple of 120 and 14 is 840; therefore 840 is the least common multiple of the numbers 8, 10, 12 and 14.

LEMMA 4.

A PROBLEM.

173. *Let* a *and* b *be any two unequal quantities whereof* c *and* d *are multiples respectively: It is required to find as many more multiples as we please of the same quantities with the same difference, that is, that* a's *multiple may always exceed that of* b *by* $\overline{c-d}$, *if* c *be greater than* d; *or else that* b's *multiple may always exceed that of* a *by* $\overline{d-c}$, *if* d *be greater than* c.

SOLUTION.

Let $c+e$ and $d+e$ represent any two multiples of this kind, that is, whose difference is $c-d$ or $d-c$; then since both c and $c+e$ are multiples of a, e will also be a multiple of a, by art. 169; for the same reason, since both d and $d+e$ are multiples of b, e will be a multiple of b; therefore *If* e *be any common multiple of* a *and* b, $c+e$ *and* $d+e$ *will be multiples of the same, and such as were required and not otherwise. Let now* e *be the least common multiple of* a *and* b; *then it is plain that* $c+e$ *and* $d+e$ *will be the next in their kind greater than* c *and* d, *and that* $c+2e$ *and* $d+2e$ *will be the next greater than* $c+e$ *and* $d+e$, *and so on* ad infinitum: *on the other hand* $c-e$ *and* $d-e$ *will be the next in their kind less than* c *and* d, *and* $c-2e$ *and* $d-2e$ *the next less than* $c-e$ *and* $d-e$, *and so on, so long as the subtraction can be continued.*

Whence it follows, that *If* e *be the least common multiple of* a *and* b, *and if either* c *or* d *or both be less than* e, *those multiples* c *and* d *will be the least in their kind:* as for example, let $a=4$ and $b=6$, then will 8 and 18 be two multiples of a and b, whose difference is 10; and they will be the least in their kind, because the least common multiple of a and b is 12, and 8 is less than 12: but if other multiples with the same difference be required, they will easily be had by a continual addition of the least common multiple 12; as 20 and 30, 32 and 42, 44 and 54, &c *ad infinitum.*

LEMMA 5.

LEMMA 5.

A THEOREM.

174. *Let* a *and* b *be any two quantities whose greatest common measure is* c: *I say then that if any two unequal multiples of* a *and* b *be taken and compared, their difference can never be less than the greatest common measure* c.

For since, by the supposition, c measures both a and b, it must not only measure all their multiples, but also the differences of those multiples; and therefore no difference can be less than c. *Q. E. D.*

LEMMA 6.

A PROBLEM.

175. *It is required, having given two numbers* a *and* b, *whereof* a *is supposed to be the greater, to find two unequal multiples of these numbers whose difference shall be the least possible, that is by the last article, whose difference shall be the greatest common measure of* a *and* b.

SOLUTION.

Let the greater number be 270, and the less 112, whose greatest common measure found by art. 21, is 2: it is required then to find a multiple of 270, and another of 112, whose difference shall be 2, and that, whichsoever of the two multiples is intended for the greater. Here $a=270$, and $b=112$, which two equations I put down one under the other in the form following:

$$\text{1st, } 1a - 0b = +270,$$
$$\text{2d, } 0a - 1b = -112,$$

where $0a$ and $-0b$ are only put down to make these equations of the same form with others, which I derive from them thus: first I divide 270 the absolute term of the first equation by 112 the absolute term of the second, without having regard to it's sign, and the nearest quotient too little is 2; therefore I multiply the second equation by 2, and the product is $0a - 2b = -224$; to this equation I add the first, $1a - 0b = +270$, and the sum is $a - 2b = +46$, which I put down under the other two for a third equation, as you will find in the table annexed: this done, I again divide 112 the absolute term of the second equation, without regarding it's sign, by 46 the absolute term of the third, and the nearest quotient too little is 2; therefore I multiply the third equation by 2, and the product is $2a - 4b = +92$; to this I add the

the

the equation next above it, *viz.* $0a-1b=-112$, and the sum makes a fourth equation, to wit, $2a-5b=-20$, which I put down under the rest: then I divide 46 the absolute term of the third equation by 20 the absolute term of the fourth, and still the nearest quotient too little is 2; therefore to twice the fourth equation I add the third, and so have a fifth, to wit, $5a-12b=+6$: this being put down, I divide 20 the absolute term of the fourth equation by 6 the absolute term of the fifth, and the nearest quotient too little is 3; therefore to thrice the fifth equation I add the fourth, and so have a sixth, to wit, $17a-41b=-2$: lastly I divide 6 the absolute term of the fifth equation by 2 the absolute term of the sixth, and the true quotient is 3, but the nearest too little is 2: now if to three times the sixth equation I should add the the fifth, I should have $56a-135b=0$; but because I do not want an equation whose absolute term is nothing, but only an equation whose absolute term is the least possible, instead of the true quotient 3 I use 2 the nearest quotient too little, as I have all along hitherto done, and so adding the fifth equation to twice the sixth, I have a seventh, to wit, $39a-94b=+2$.

Having thus finished my series of equations, it is evident that the two last equations will solve the problem; for if the multiple of a is to be greater than that of b, $39a$ and $94b$ will be the multiples required, because $39a-94b=+2$; but if the multiple of b is intended to be greater than that of a, then $17a$ and $41b$ will be the multiples sought, because $17a-41b=-2$, and consequently $41b-17a=+2$. This will further be confirmed upon tryal; for if a be 270, and b 112, we shall have $39a=10530$, $94b=10528$, $17a=4590$, $41b=4592$.

Equ.	1,	$1a-0b=+270.$
	2,	$0a-1b=-112.$
	3,	$a-2b=+46.$
	4,	$2a-5b=-20.$
	5,	$5a-12b=+6.$
	6,	$17a-41b=-2.$
	7,	$39a-94b=+2.$

Lastly, that this series of absolute terms in the equations above described will always terminate at last in the greatest common measure of a and b, will be evident from the very method of finding the greatest common measure, laid down in art. 21, which is the very same with that whereby the absolute terms are found: for to find the greatest common measure of the numbers 270 and 112, 270 must be divided by 112, 112 by 46, 46 by 20, 20 by 6, and 6 by 2, and the series 270, 112, 46, 20, 6, 2 terminates in the greatest common measure: but this series is altogether the same with the series of absolute terms found above, except

that

that there, the greateſt common meaſure is twice repeated; which ariſes from hence, that the true quotient at laſt was not taken, but only the neareſt quotient too little; therefore one of the two laſt equations will always ſolve the problem. *Q. E. D.*

COROLLARY 1.

Since the fraction $\frac{a}{b}$ or $\frac{270}{112}$, when reduced to it's leaſt terms, is $\frac{135}{56}$, it follows by art. 171, that $56a=135b$ is the leaſt common multiple of a and b, and conſequently that $56a-135b=0$ is the ſimpleſt equation that can have 0 for it's abſolute term. Let now this equation be continually added to either of the two laſt equations which ſolves the problem, and you will have an infinite number of ſolutions more of the ſame problem, that is, you will have an infinite number of equations whoſe abſolute terms are + or −2: thus if to the laſt equation $39a-94b=2$ be added the equation $56a-135b=0$, you will have $95a-229b=2$; and if again be added the ſame equation, you will have $151a-364b=2$, and ſo on: again, if to the laſt equation but one, to wit, $17a-41b=-2$ be added the equation $56a-135b=0$, you will have $73a-176b=-2$; and if again be added the ſame equation, you will have $129a-311b=-2$, and ſo on *ad infinitum*.

COROLLARY 2.

The abſolute terms of the equations above deſcribed will always be affirmative and negative alternately, and the abſolute term of the laſt equation will be negative or affirmative according as the number of equations is even or odd: thus in the foregoing table the abſolute terms were $+270$, -112, $+46$, -20, $+6$, -2, $+2$, and the laſt term was $+2$ as poſſeſſing an odd place.

SCHOLIUM.

If the foregoing problem be ſtated algebraically, it will ſtand thus; *It is required, having given two numbers* a *and* b, *whoſe greateſt common meaſure is* c, *to find other two numbers* x *and* y *of ſuch a nature, that being multiplied by the numbers* a *and* b *reſpectively, the difference of the products* ax − by *may be found equal to + or* − c.

Here then we have but one equation for finding two unknown quantities, and that is the reaſon why the problem admits of an infinite number of ſolutions, to wit, by a continual addition of the equation $56a-135b=0$, as in corollary 1: but though this equation may be continually added to any of the equations exhibited in the foregoing table, and that *ad infinitum*, yet it cannot be ſubtracted from any of them; which

is

is a manifest argument that those equations are the most simple of their kind, that is, the most simple of all others that have the same absolute terms; see art. 173: this will be more universally demonstrated in art. 177. In the mean time, in order to form general conclusions concerning the nature of those equations, it will be proper in all the equations except the two first, to denote the coefficients of a and b at least, in general terms, thus: in the third equation we have $1a-2b=+46$; make $p=1$ and $P=2$, and you will have $pa-Pb=+46$: in like manner in the fourth equation, if q be made equal to 2 and Q equal to 5, you will have $qa-Qb=-20$; and so on. See the equations according to this way of notation:

Equ. 1, $1a-0b=+270.$
2, $0a-1b=-112.$
3, $pa-Pb=+46.$
4, $qa-Qb=-20.$
5, $ra-Rb=+6.$
6, $sa-Sb=-2.$
7, $ta-Tb=+2.$

LEMMA 7.

A THEOREM.

176. *If in any of the equations exhibited in the last article, the coefficients of* a *and* b *be multiplied crosswise into the coefficients of* a *and* b *in the equation next it, whether above or below,* b's *coefficient being considered as affirmative; I say then that the difference of the two products thence arising will always be equal to unity.*

As for example, let the two last equations be taken, and let t the coefficient of a in the last equation be multiplied into S the coefficient of b in the last but one; let also T the coefficient of b in the last equation be multiplied into s the coefficient of a in the last but one: I say then that the difference betwixt the two products tS and Ts will be equal to unity; and that the same will happen throughout the whole series, provided that the equations made choice of be such as lie nearest one another.

To demonstrate this, let the absolute term of the last equation but two be divided by the absolute term of the last but one, and let the nearest quotient too little be n; then it is plain from the nature of these equations, that $nsa+ra$ will be equal to ta, and that $-nSb-Rb$ will be equal to $-Tb$; therefore $ns+r=t$, and $nS+R=T$. Put now instead of t and T their values thus found, and you will have $tS=nSs+Sr$, and $Ts=nSs+sR$; therefore the difference between tS and Ts will be the same with the difference between sR and Sr; and for the

the ſame reaſon, the difference between sR and Sr is the ſame with the difference between rQ and Rq; and this again the ſame with the difference between qP and Qp, &c: therefore if this difference in any two neareſt equations be known, it will every where be known, as being every where the ſame; but 1 the coefficient of a in the firſt equation being multiplied into 1 the affirmative coefficient of b in the ſecond, gives 1 for the product; and moreover 0 the coefficient of b in the firſt equation being multiplied into 0 the coefficient of a in the ſecond gives 0 for the product, and the difference betwixt the products 1 and 0 is 1; therefore the difference of the products every where will be 1. *Q. E. D.*

LEMMA 8.

A THEOREM.

177. *Suppoſing all things as in the two laſt articles, if the coefficients of* a *and* b *in the ſeveral equations exhibited in the ſcholium to the hundred ſeventyfifth article be turned into fractions, by making the coefficients of* b *the numerators, and thoſe of* a *the denominators; I ſay then that the fractions* $\frac{0}{1}$, $\frac{1}{0}$, $\frac{P}{p}$, $\frac{Q}{q}$, $\frac{R}{r}$, $\frac{S}{s}$, $\frac{T}{t}$ *will all be in their leaſt terms.*

For if the fraction $\frac{T}{t}$ for inſtance, be not in it's leaſt terms, the numbers T and t muſt admit of ſome common meaſure greater than unity: be it ſo, and let this common meaſure be called m; then ſince the number m meaſures both the numbers T and t, it muſt alſo meaſure both the products Ts and tS, and conſequently their difference; but the difference of theſe two products is unity, by the laſt article; therefore m a number greater than unity muſt meaſure unity, which is abſurd; therefore the fraction $\frac{T}{t}$ is in it's leaſt terms: and the ſame may be affirmed of all the reſt. *Q. E. D.*

COROLLARY.

If the abſolute term of the laſt equation but one, which is the greateſt common meaſure of a and b, be divided by the abſolute term of the laſt, which is alſo the ſame common meaſure, the quotient will be 1: therefore if the laſt equation be multiplied by 1, and to the product be added the laſt equation but one, or which is the ſame thing, if the two laſt equations be added together, there will ariſe an equation of this form, $va - Vb = 0$, the two abſolute terms deſtroying each other;

other; whence $\frac{V}{v}=\frac{a}{b}$; but the fraction $\frac{V}{v}$ is in it's least terms by this proposition, as being produced after the same manner as the rest; therefore $av=bV$ is the least common multiple of a and b, and the equation $av-bV=0$ is the simplest that can have 0 for it's absolute term: but av and bV are greater multiples of a and b than can be found in any of the equations exhibited in art. 175, as is evident from the *genesis* of those equations; therefore the equation $av-bV=0$ cannot be subtracted from any of those equations, at least without destroying the coefficient of a; therefore those equations are the simplest of their kind.

LEMMA 9.

A THEOREM.

178. *Supposing all things as in the last article, I say that the fractions* $\frac{0}{1}, \frac{1}{0}, \frac{P}{p}, \frac{Q}{q}, \frac{R}{r}, \frac{S}{s}, \frac{T}{t}$ *are of such a nature as to be alternately less and greater than the fraction* $\frac{a}{b}$, *but yet so as constantly to converge towards it.*

Thus the fraction $\frac{0}{1}$ is infinitely less than the fraction $\frac{a}{b}$, and the fraction $\frac{1}{0}$ is infinitely greater; for the less the denominator of any fraction is, *cæteris paribus*, the greater will be the fraction; therefore if the denominator be nothing, or infinitely small, the value of such a fraction must be infinitely great: thus again the fraction $\frac{P}{p}$ is less than $\frac{a}{b}$, and the fraction $\frac{Q}{q}$ greater, and so on alternately. I say further, that the fraction $\frac{Q}{q}$ approaches nearer to $\frac{a}{b}$ than doth the fraction $\frac{P}{p}$, and the fraction $\frac{R}{r}$ nearer than $\frac{Q}{q}$, and so on.

To demonstrate this, let the equation $ca-Cb=\pm d$ be a general representative of all the equations in art. 175; and dividing both sides by bc, we shall have $\frac{a}{b}-\frac{C}{c}=\pm\frac{d}{bc}$; whence I infer 1st, that the fraction $\frac{a}{b}$ will be greater or less than the fraction $\frac{C}{c}$, according as the abso-

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lute term d is affirmative or negative; 2*dly*, that the leſs the abſolute term d is, or the greater the denominator c is, the nearer will the fraction $\frac{C}{c}$ approach towards the fraction $\frac{a}{b}$: but throughout the whole ſeries of equations in art. 175, the abſolute terms are alternately affirmative and negative; therefore the ſeries of fractions deduced from thoſe equations will be of ſuch a nature, that the fractions will be alternately leſs and greater than the fraction $\frac{a}{b}$: it is alſo evident from the nature of the equations above-mentioned, that the ſeries of terms repreſented by d will be conſtantly diminiſhing, and that the terms repreſented by c will be conſtantly increaſing; and therefore upon both theſe accounts, the fractions repreſented by $\frac{C}{c}$ muſt converge on both ſides towards the fraction $\frac{a}{b}$.

COROLLARY.

Therefore if any two fractions that are neareſt to each other be taken, as $\frac{T}{t}$ and $\frac{S}{s}$, the fraction $\frac{a}{b}$ muſt lie between them, but ſo as to lie nearer to the fraction $\frac{T}{t}$ than to the fraction $\frac{S}{s}$, otherwiſe the ſeries would not converge.

LEMMA 10.

A THEOREM.

179. *Suppoſing all things as in the foregoing articles, I ſay that the fractions* $\frac{\mathrm{P}}{\mathrm{p}}, \frac{\mathrm{Q}}{\mathrm{q}}, \frac{\mathrm{R}}{\mathrm{r}}, \frac{\mathrm{S}}{\mathrm{s}}, \frac{\mathrm{T}}{\mathrm{t}}$ *exhibit the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *more accurately than any other fractions whatever of leſs denominations can do; and particularly I ſay that the fraction* $\frac{\mathrm{T}}{\mathrm{t}}$ *approaches nearer to the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *than any fraction can do whoſe denominator is leſs than* t.

If this be denied, let $\frac{C}{c}$ be a fraction whoſe denominator c is leſs than t: then I affirm that $\frac{C}{c}$ cannot be equal to $\frac{T}{t}$; for the fraction $\frac{T}{t}$ is in it's leaſt terms already, by art. 177; therefore if the fraction $\frac{C}{c}$

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was equal to the fraction $\frac{T}{t}$, the denominator c muſt be equal to, or ſome multiple of the denominator t, whereas c is leſs than t by the ſuppoſition: now ſince the fraction $\frac{a}{b}$ lies between the two fractions $\frac{T}{t}$ and $\frac{S}{s}$, but nearer to $\frac{T}{t}$ by the laſt article, it follows that if the fraction $\frac{C}{c}$ approached nearer to $\frac{a}{b}$ than doth the fraction $\frac{T}{t}$, it muſt lie alſo between $\frac{T}{t}$ and $\frac{S}{s}$; but the difference between $\frac{T}{t}$ and $\frac{S}{s}$ is $\frac{1}{ts}$, by art. 176; and the difference between $\frac{C}{c}$ and $\frac{S}{s}$ is $\frac{Cs-cS}{cs}$, or $\frac{cS-Cs}{cs}$; and this difference cannot be leſs than $\frac{1}{cs}$: ſince then $\frac{1}{cs}$ is greater than $\frac{1}{ts}$ becauſe c is leſs than t, it follows that the difference between $\frac{S}{s}$ and $\frac{C}{c}$ is greater than the difference between $\frac{S}{s}$ and $\frac{T}{t}$; therefore the fraction $\frac{C}{c}$ cannot lie between the two fractions $\frac{S}{s}$ and $\frac{T}{t}$, and conſequently cannot approach ſo near the fraction $\frac{a}{b}$ as doth the fraction $\frac{T}{t}$. *Q. E. D.*

SCHOLIUM I.

In computing the equations in the 175th article, the quotients uſed for that purpoſe were 2, 2, 2, 3, 2, the laſt quotient 3 being diminiſhed by unity to give ſuch a ſolution of the problem there conſidered as was neceſſary to anſwer the end for which it was propoſed; but had the true quotient 3 been taken, the laſt equation would have been $56a-135b=0$, as was there obſerved; and the fraction derived from it would have been $\frac{135}{56}$, which is nothing elſe but the fraction $\frac{a}{b}$ in it's leaſt terms; therefore if the laſt quotient be taken true, the fractions derived from thoſe equations, to wit, $\frac{2}{1}$, $\frac{5}{2}$, $\frac{12}{5}$, $\frac{41}{17}$, $\frac{135}{56}$ will not only converge on both ſides towards the intermediate fraction $\frac{a}{b}$, but at laſt will actually terminate in it.

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As to the practice of computing these fractions, there is no necessity of introducing the terms a and b into the series, and much less of computing the equations in order to deduce the fractions from them; for as the terms of these fractions are nothing else but the coefficients of a and b, they may be obtained much easier by themselves, in the manner following: Let the fraction $\frac{270}{112}$ be proposed, whose numerator and denominator are the terms a and b, as in art. 175, and let it be required to find other fractions of lesser denominations which, though not equal to the fraction proposed, shall nevertheless come as near it as any particular purpose requires, and nearer than any other fraction of a less denomination than these can: here the greater number 270 divided by the less 112 quotes 2, and there remains 46; again, 112 divided by 46 quotes 2, and there remains 20; 46 divided by 20 quotes 2, and there remains 6; 20 divided by 6 quotes 3, and there remains 2; and lastly 6 divided by 2 quotes 3, and there remains 0; therefore the quotients arising from this continual division are 2, 2, 2, 3, 3. These quotients being thus obtained, in the next place I assume two fractions $\frac{0}{1}$ and $\frac{1}{0}$ to begin with, and multiplying 1 and 0, the terms of the latter fraction, by the first quotient 2, the products are 2 and 0, to which I add 0 and 1, the terms of the first fraction, and so have 2 and 1 for the terms of my first finite fraction; therefore my first finite fraction is $\frac{2}{1}$; these terms 2 and 1 I multiply by my second quotient 2, and the products are 4 and 2, to which adding 1 and 0, the terms of the fraction immediately going before, I have $\frac{5}{2}$ for my next fraction; these terms 5 and 2 I multiply by my third quotient 2, and the products are 10 and 4, which with 2 and 1, the terms of the fraction immediately before it, give $\frac{12}{5}$ for my next fraction; these terms 12 and 5 I multiply by my fourth quotient 3, and the products are 36 and 15, which with 5 and 2, the terms of the next antecedent fraction, give $\frac{41}{17}$ for the terms of my next fraction; lastly I multiply 41 and 17 by my last quotient 3, and the products are 123 and 51, which with 12 and 5, the terms of the fraction before it, give $\frac{135}{56}$ for my last fraction; and so the fractions thus obtained are $\frac{2}{1}$, $\frac{5}{2}$, $\frac{12}{5}$, $\frac{41}{17}$, $\frac{135}{56}$. Where it may be observed, that if

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I had placed the fraction $\frac{1}{0}$ before the fraction $\frac{0}{1}$, and then had derived the rest from these as before, the fractions would have come out inverted, to wit, $\frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{17}{41}, \frac{56}{135}$; and these fractions would have converged towards, and at last would have terminated in the intermediate one $\frac{b}{a}$, with this difference however, that now the first fraction $\frac{1}{2}$ would have been greater than $\frac{b}{a}$, the next $\frac{2}{5}$ less, and so on alternately.

The better to see the approximation of these fractions, I reduce them all to other fractions whose common numerator is 270, thus: as 2 the numerator of the first fraction is to 1 the denominator, so is 270 the new numerator of the same fraction to 135 the new denominator; whence $\frac{2}{1} = \frac{270}{135}$, which differs no more from $\frac{270}{112}$ than might be expected, considering the simplicity of the fraction $\frac{2}{1}$: again, as 5 the numerator of the next fraction is to 2 the denominator, so is 270 to 108; therefore $\frac{5}{2} = \frac{270}{108}$, which differs much less from $\frac{270}{112}$ than did the former fraction $\frac{2}{1}$: 3dly, as 12 the numerator of the next fraction is to 5 the denominator, so is 270 to $112\frac{1}{2}$; therefore $\frac{12}{5} = \frac{270}{112\frac{1}{2}}$, which is very near $\frac{270}{112}$: 4thly, as 41 the numerator of the next fraction is to 17 the denominator, so is 270 to $111\frac{39}{41}$ or $112 - \frac{1}{20}$ nearly; therefore $\frac{41}{17} = \frac{270}{112 - \frac{1}{20}}$: lastly, as 135 the numerator of the last fraction is to 56 the denominator, so is 270 to 112; therefore $\frac{135}{56} = \frac{270}{112}$.

For another example of this sort of approximation, take the proportion of the circumference of a circle to it's diameter, which according to *Van Ceulen*'s numbers when abbridged, is that of 314159265359 to 100000000000. Here making $a = 314159265359$, and $b = 100000000000$, let it be required to find a fraction in more simple terms that shall nearly express the fraction $\frac{a}{b}$, that is, more nearly than any other fraction of a less

leſs denomination; and the proceſs will be as follows: 1ſt, a divided by b quotes 3, and there remains $c=14159265359$: 2dly, b divided by c quotes 7, and there remains $d=885142487$: 3dly, c divided by d quotes 15, and there remains $e=882128054$: 4thly, d divided by e quotes 1, and there remains $f=3014433$: 5thly, e divided by f quotes 292, and there remains $g=1913618$. This laſt quotient 292 being ſo very large, I conclude that the four firſt quotients 3, 7, 15 and 1 will give the laſt fraction accurately enough, ſince it's terms muſt be multiplied by no leſs a number than 292 to obtain another that is more ſo. Uſing then the four firſt quotients 3, 7, 15, 1, I proceed as in the foregoing example, and ſo find four fractions $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$ and $\frac{355}{113}$, which laſt fraction will be found exact enough for all purpoſes; for ſince the circumference of a circle is to the diameter as a to b, and the fraction $\frac{a}{b}$ is nearly equal to $\frac{355}{113}$, it follows that the circumference is to the diameter as 355 to 113 nearly, or as $\frac{355}{113}$ is to 1, or (reducing the fraction to decimals) as 3.1415929 to 1, or laſtly as 31415929 to 10000000, the antecedent of which proportion is true to 7, and almoſt to 8 places; for the proportion ought to be as 31415927 to 10000000.

Artiſts for the better remembering the proportion betwixt the diameter and circumference of a circle, as we have here eſtabliſhed it, *viz.* 113 to 355, are uſed to conceive it thus: put down the three firſt odd numbers 1, 3 and 5, writing down every number twice, or at leaſt imagine them to be put down thus, 113355; then will the three firſt numbers 113 belong to the diameter, and the three laſt, to wit, 355 to the circumference.

SCHOLIUM 2.

The fractions treated of in the laſt ſcholium may be called primary or principal fractions, all converging to, and at laſt meeting in the intermediate fraction $\frac{a}{b}$; but if it be required that the ſeries be compleat, other fractions called ſecondary ones muſt be inſerted between the primary ones, when the quotient is greater than unity, by changing the multiplication in the laſt ſcholium into a continual addition: in order to do this, it will be proper to make two rows of fractions, the upper to contain all the fractions that are greater than $\frac{a}{b}$, and the lower

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to contain all those that are less, so as that $\frac{1}{0}$ may be the first fraction in the upper row, and $\frac{0}{1}$ the first in the lower; then making use of the first example in the foregoing scholium, where the quotients were 2, 2, 2, 3, 3, because the equations in the 175th article were particularly adapted to that case, add the terms of the fraction $\frac{1}{0}$ twice to the terms of the fraction $\frac{0}{1}$, because the first quotient was 2, and you will have two fractions arising from that continual addition, to wit, $\frac{1}{1}$ and $\frac{2}{1}$ to be placed in the lower row: add the terms of this last fraction $\frac{2}{1}$ twice to the terms of the fraction $\frac{1}{0}$, because the second quotient was 2, and you will have two fractions $\frac{3}{1}$ and $\frac{5}{2}$ to be placed in the upper row: add the terms of this last fraction $\frac{5}{2}$ twice to the terms of the last fraction $\frac{2}{1}$ in the lower row, because the third quotient was 2, and you will have the fractions $\frac{7}{3}$ and $\frac{12}{5}$ to be placed in the lower row: add the terms of this last fraction $\frac{12}{5}$ thrice to the terms of the last fraction $\frac{5}{2}$ in the upper row, because the fourth quotient was 3, and you will have $\frac{17}{7}$, $\frac{29}{12}$ and $\frac{41}{17}$ to be placed in the upper row: add the terms of this last fraction $\frac{41}{17}$ thrice to the last fraction $\frac{12}{5}$ in the lower row, because the last quotient was 3, and you will have $\frac{53}{22}$, $\frac{94}{39}$ and $\frac{135}{56}$ to be placed in the lower row: and thus you will have finished the whole series, or rather two serieses of fractions, whereof the upper series consists of fractions greater than $\frac{a}{b}$, and the lower of fractions that are less, as appears by the following table:

Fractions

Fractions greater than the true, $\frac{1}{0}$ $\frac{3}{1}$ $\frac{5}{2}$ $\frac{17}{7}$ $\frac{29}{12}$ $\frac{41}{17}$.

Fractions leſs than the true, $\frac{0}{1}$ $\frac{1}{1}$ $\frac{2}{1}$ $\frac{7}{3}$ $\frac{12}{5}$ $\frac{53}{22}$ $\frac{94}{39}$ $\frac{135}{56}$.

The laſt fraction in the lower row, to wit $\frac{135}{56}$, is equal to $\frac{a}{b}$, and therefore, properly ſpeaking, belongs to neither row: every fraction in the upper ſeries approaches nearer to $\frac{a}{b}$ than any other fraction in ſimpler terms can do that is greater than $\frac{a}{b}$, and every fraction in the lower ſeries approaches nearer it than any other in ſimpler terms can do that is leſs than $\frac{a}{b}$; but it ſometimes happens that a fraction on one ſide ſhall not approach ſo near it as a fraction on the other, though in more ſimple terms: thus the fraction $\frac{17}{7}$ in the upper ſeries differs more from $\frac{270}{112}$ or $\frac{a}{b}$ on one ſide, than $\frac{12}{5}$ doth on the other; for the exceſs of $\frac{17}{7}$ above $\frac{270}{112}$ is $\frac{1}{56}$ or $\frac{10}{560}$, whereas the exceſs of $\frac{270}{112}$ above $\frac{12}{5}$ is $\frac{6}{560}$; but this never happens when the fraction that hath the greater denominator is a primary fraction.

All I have delivered in this ſcholium will beſt be comprehended by treating the equations in the 175th article after the ſame manner as the fractions here, that is, by turning the multiplication there made uſe of into a continual addition; but for the better diſtinguiſhing theſe equations, make two columns, placing all the equations whoſe abſolute terms are negative in the left-hand column, and all thoſe whoſe abſolute terms are affirmative on the right, ſo as that the firſt equation on the left hand may be $0a - 1b = -112$, and the firſt on the right hand may be $1a - 0b = +270$: this done, you may proceed as follows; 1ſt, add the equation $0a - 1b = -112$ twice to the equation $1a - 0b = +270$, and you will have two equations to be placed in the right-hand column, to wit, $1a - 1b = +158$, and $1a - 2b = +46$: 2dly, add this laſt equation $1a - 2b = +46$ twice to the equation $0a - 1b = -112$, and you will have two equations to be placed in the left-hand column, to wit, $1a - 3b = -66$, and $2a - 5b = -20$: 3dly, add this laſt equation twice to the equation $1a - 2b = +46$, and you will have two equations to be placed on the right hand, to wit, $3a - 7b = +26$,

$-7b=+26$, and $5a-12b=+6$: 4thly, add this laſt equation thrice to the equation $2a-5b=-20$, and you will have three equations to be placed on the left hand, to wit, $7a-17b=-14$, $12a-29b=-8$, and $17a-41b=-2$: 5thly, add this laſt equation thrice to the equation $5a-12b=+6$, and you will have three equations to be placed in the right-hand column, to wit, $22a-53b=+4$, $39a-94b=+2$, and $56a-135b=0$. Thus you will have finiſhed your ſerieſes of equations, from whence the two ſerieſes of fractions already computed have been derived.

$$0a-1b=-112.$$
$$1a-3b=-66.$$
$$2a-5b=-20.$$
$$7a-17b=-14.$$
$$12a-29b=-8.$$
$$17a-41b=-2.$$

$$1a-0b=+270.$$
$$1a-1b=+158.$$
$$1a-2b=+46.$$
$$3a-7b=+26.$$
$$5a-12b=+6.$$
$$22a-53b=+4.$$
$$39a-94b=+2.$$
$$56a-135b=0.$$

Much more might be produced upon this head; but I fear moſt of my readers think I have been too prolix already, eſpecially thoſe, who being leſs converſant in numbers, cannot ſo eaſily be made ſenſible of the uſefulneſs of this doctrine.

This way of approximation to fractions of higher denominations is owing to the ſagacity of the ingenious Mr. *Cotes* late Profeſſor of Aſtronomy and experimental Philoſophy in this Univerſity, whoſe great and happy genius to all mathematical enquiries made him conceive and treat the moſt difficult ſubjects with the utmoſt ſimplicity and eaſe; and whoſe poſthumous pieces not long ago publiſhed by his worthy ſucceſſor Dr. *Smith*, will ever be eſteemed and admired, ſo long as a good taſte and ſound judgment ſhall have any being amongſt mankind.

This ſubject had been handled before, though in a very different way, by the learned Dr. *Wallis*, and Monſieur *Huygens*; but I am of opinion, that whoever makes the compariſon will eaſily agree with me, that this method is much readier and leſs embarraſſed than either of their's.

LEMMA II.

A THEOREM.

180. *Let there be two fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$ *ſuch, that if a continual diviſion be made from* a *and* b, *according to the method for finding the greateſt*

greatest common measure, the quotients thence arising shall be found to be the same, both in quantity, order and number, with the quotients arising from a continual division made from c *and* d*: I say then that the fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$ *will be equal one to the other.*

To demonstrate this, let the quotients in both cases be 1, 2 and 3; and by the help of these quotients let a converging series of primary fractions be computed, as in the first scholium to the last article, to wit, $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{10}{7}$; then it is plain that this series will equally relate to both fractions, will equally converge to both, and at last terminate in both; therefore the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ must be equal one to the other, since they are both equal to the last fraction of the series, to wit, $\frac{10}{7}$. *Q. E. D.*

COROLLARY.

Since the two fractions mentioned in this article, to wit, $\frac{a}{b}$ and $\frac{c}{d}$ will be equal one to the other, whether the number of divisions and quotients be few or many, it is plain that the nature of this equality does not depend upon the supposition of a last division, though the demonstration does; and therefore this article will be true, even though the number of quotients be infinite.

SCHOLIUM.

If a, b, c and d, instead of numbers, represent any other quantities whatever, and the quotients from a and b be every way the same with the quotients from c and d, then the sense of the foregoing lemma is, that a will have the same proportion to b that c hath to d; and that, whether the quantities a, b, c, d be commensurable or incommensurable: and therefore if we lay down this as a definition of proportionality, to wit, that *Four quantities are said to be proportionable, when the quotients of a continual division from the two first are every way the same with the quotients of a continual division from the two last*; this definition will reach incommensurables, and will sufficiently justify our application of proportion to such quantities: for now all the difference betwixt the proportion of commensurable and incommensurable quantities will only be this, that whereas in the case of commensurables, the quotients will break off, in the case of incommensurables they will run on

ad infinitum; but the quotients from the two first will be every way the same with the quotients from the two last, which sufficiently marks out their proportionality, and is, as I take it, the most distinct definition that can be given of proportionality, though not so useful as that in the fifth definition to the fifth book of the elements, because the other properties of proportion are not so easily deducible from it.

OF INCOMMENSURABLES.

An apology for introducing the doctrine of Incommensurables into this place.

181. I doubt not but it will be thought by some a little out of method to introduce the doctrine of incommensurables into this place; but my first problem ought by no means to be rejected, whether we consider the elegancy of the problem itself, or the close connexion it has with some of the foregoing articles, which it may serve further to illustrate; and as this problem relates purely to incommensurables, without a previous knowledge whereof it cannot possibly be understood, I thought this a proper place to bring my reader into some acquaintance with them; and if, after I found myself obliged to dip into this subject, I drew out from thence somewhat more than my present occasion required, it was because I easily foresaw that the rest would be of no less use on other occasions. In short, this doctrine of incommensurables is certainly very entertaining, and a curious instance of most subtil and refined reasoning, and therefore cannot, if closely and clearly delivered, but be acceptable to every ingenuous reader, who is really more inquisitive after true and substantial knowledge than after matter of criticism and censure; and this I hope will be sufficient to justify me.

I shall insert here but a very small part of what *Euclid* has delivered more at large upon this head; and my method will be, first to demonstrate some properties naturally flowing from the idea of incommensurability, and from one another, and then to find a subject for these properties, that is, to demonstrate that there are actually existing in nature such things as incommensurable quantities.

N. B. If any one has a mind to satisfy himself concerning the existence of incommensurables without being at the pains of reading all I have written upon the subject, he may first read the hundred eighty second article, and then pass over to the tenth proposition and those that follow, which are so contrived as to have little or no dependence upon the rest.

DEFI-

DEFINITIONS AND AXIOMS.

182. Def. 1ſt. *One quantity is ſaid to meaſure another, when being taken as often as poſſible out of that other, there is nothing left.*

2d. *When a quantity will meaſure two others, it is called their common meaſure.*

3d. *And quantities are ſaid to be commenſurable or incommenſurable, according as they will or will not admit of a common meaſure.*

4th. *Whole numbers are ſaid to be prime to one another, which will admit of no common meaſure but unity.*

5th. *A ſquare number, in the ſenſe we are here to uſe it, is any whole number that will admit of itſelf, or ſome other whole number for it's ſquare root, excluſive of all fractions both out of the ſquare and out of the root.*

6th. *Whenever a leſs whole number meaſures a greater, the leſs is called an aliquot part of the greater, and the greater a multiple of the leſs.*

SCHOLIUM.

Fractions, as far as I can find, were very little regarded by the Ancients, eſpecially before *Archimedes*'s time; but what we now expreſs and perform by fractions, they accompliſhed by the help of proportion in whole numbers; ſo that by number or numbers with them, muſt always be underſtood whole numbers, and by aliquot parts, ſuch leſſer whole numbers as will meaſure others that are greater: thus the aliquot parts of the number 6 were 1, 2 and 3; but the mixt number $1\frac{1}{2}$, or the fraction $\frac{3}{2}$, though it meaſures 6 juſt four times, was nevertheleſs excluded out of the number of it's aliquot parts: and to ſay the truth, if this be not done, every number will have as many aliquot parts as we pleaſe; and ſo the diſtinction of prime and compoſite numbers, and the diſtinction of numbers that are prime and not prime to each other, will be utterly loſt; our idea of proportionality in numbers will be rendered obſcure and confuſed; and that excellent treatiſe of numbers by *Euclid* in his ſeventh, eighth and ninth books, will become in a great meaſure uſeleſs. Not that I would here be underſtood as if I did not think fractions exceeding uſeful in all arithmetical operations; ſo far am I from thinking otherwiſe, that I have frequently uſed them myſelf in caſes where others uſed proportion, purely becauſe I looked upon fractions as more tractable: but what I would chiefly ſignify by the foregoing diſcourſe is, that whether fractions be admitted into arithmetical operations or not, the doctrine of whole numbers being previous to, and diſtinct from that of fractions, the Ancients were very juſtifiable in treating it diſtinctly by itſelf, without any regard to fractions.

COROLLARIES FROM THE DEFINITIONS.

1ſt. *All whole numbers are commenſurable:* for unity is a common meaſure to them all.

2d. *All fractions are commenſurable:* for if the fractions $\frac{a}{b}$ and $\frac{c}{d}$ be reduced to the ſame denomination, they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$; and $\frac{1}{bd}$ will meaſure them both.

3d. *All whole numbers and mixt numbers compared together, all mixt numbers compared with one another, all whole numbers and fractions, all mixt numbers and fractions are commenſurable:* for all mixt numbers may be reduced to fractions; and all whole numbers may be conſidered as fractions whoſe denominators are units.

Axiom 1. *If one quantity meaſures a ſecond, and that a third, the firſt quantity will alſo meaſure the third.*

2d. *Whatever quantity meaſures two others, the ſame will meaſure both their ſum and their difference.*

3d. *Two finite homogeneous quantities can never be ſo different one from another, but that the leſſer may be multiplied till it exceeds the greater.*

4th. *Equal fractions may be converted into equal proportions in whole numbers, by making the numerators antecedents, and the denominators conſequents; and* vice verſa, *equal proportionals in whole numbers may be changed into equal fractions.*

PROPOSITION I.

183. *All commenſurable quantities are to one another as number to number; and* vice verſa, *all quantities that are to one another as number to number are commenſurable.*

For firſt, let a and b be two commenſurable quantities; I ſay then that the proportion of a to b, let it be what it will, may be expreſſed in whole numbers: for as a and b are commenſurable quantities, let c meaſure them both; let c meaſure a juſt three times, and b four times; then will $3c = a$, and $4c = b$, and a will be to b as 3 to 4. *Q. E. D.*

Again, let a and b be two ſuch quantities that the proportion of a to b may be expreſſed in whole numbers; as for example, let a be to b as 3 to 4, and let c be a third part of a; then will $3c = a$, and $4c = b$, and ſo the quantity c will meaſure both a and b; therefore the quantities a and b will be commenſurable. *Q. E. D.*

PROPO-

PROPOSITION 2.

184. *Incommensurable quantities are not to each other as number to number; and* vice versa, *quantities that are not to each other as number to number are incommensurable.*

For first, let a and b be two incommensurable quantities; I say then that the proportion of a to b cannot possibly be expressed in whole numbers: for if it could, those quantities a and b would be commensurable, by the latter part of the foregoing proposition, which contradicts the supposition. In the next place, let the quantities a and b be such, that the proportion of a to b cannot possibly be expressed in whole numbers; I say then that those quantities a and b will be incommensurable: for if they were commensurable it might, by the former part of the foregoing proposition. *Q. E. D.*

COROLLARY.

If the proportion of incommensurable quantities be too subtil to be expressed in whole numbers, neither can it in fractions: for all fractions are commensurable, by the second corollary to the definitions, and as such, are to one another as number to number, by the first proposition.

PROPOSITION 3.

185. *If four quantities* a, b, c *and* d *be proportionable, so as that* a *is to* b *as* c *to* d; *I say then that if the two first* a *and* b *be commensurable, the two last* c *and* d *will also be commensurable; but if the two first* a *and* b *be incommensurable, the two last* c *and* d *will also be incommensurable.*

For first, let a and b be commensurable; then will a be to b as number to number, by the first proposition; but as a is to b so is c to d; therefore c will be to d as number to number; therefore c and d will be commensurable by the first proposition. *Q. E. D.*

In the next place, let a and b be incommensurable; I say then that c and d will also be incommensurable: for if c and d were commensurable, a and b would also be commensurable, by the former part of the proposition, because c is to d as a to b; but a and b are not commensurable, by the supposition; therefore neither are c and d; therefore c and d are incommensurable. *Q. E. D.*

COROLLARY.

By the demonstration of this proposition it appears, that *If incommensurability can be proved to belong to any one sort of continued quantity whatever,*

whatever, it will equally belong to all; for in this demonſtration, a and b may be quantities of any one ſort, and c and d thoſe of another. The ſeveral ſorts of continued quantity, ſuch as length, ſuperficial and ſolid ſpace, time, weight, motion &c, are all equally ſubtil in their conſtitution; and therefore if a, b, c and d be all continued quantities, there can be no abſurdity in ſuppoſing them all proportionable: but if the antecedent and conſequent of one proportion be continued quantities, and thoſe of the other diſcontinued or numbers, theſe latter being of a leſs ſubtil compoſition, cannot always be proportionable to the former, as appears from the ſecond propoſition; though on the other hand, the former may always be proportionable to the latter.

SCHOLIUM.

Hence appears the neceſſity of *Euclid's* treating proportionable magnitudes or continued quantities, which have no ſuch thing as a leaſt finite part, after a very different manner from proportionable numbers, whoſe leaſt finite part is unity.

PROPOSITION 4.

186. *If two quantities be commenſurable to a third, they will be commenſurable to one another.*

Let a and b be two quantities commenſurable to a third, which we will call c; I ſay then that the quantities a and b will be commenſurable to one another: for firſt, ſince a and c are commenſurable, they muſt be to one another as number to number, by the firſt propoſition: let then a be to c as the number d to the number e. Again, ſince b and c are ſuppoſed commenſurable, let c be to b as the number f to the number g; then ſince a is to c as d to e, or as df to ef; and again, ſince c is to b as f to g, or as ef to eg, it follows *ex æquo*, that a is to b as the number df to the number eg, and conſequently that a and b are commenſurable, by the firſt propoſition. *Q. E. D.*

COROLLARY 1.

Hence it appears that two incommenſurable quantities cannot both be commenſurable to a third; becauſe none but commenſurables can, by this propoſition.

COROLLARY 2.

If there be three homogeneous quantities a, b *and* c, *whereof* a *is incommenſurable to* b, *and* b *is commenſurable to* c; *I ſay then that* a *is incommenſurable to* c: for if it was not, we ſhould have two incomme[nſurables] *that* [bo]*th* ra

rable quantities *a* and *b* both commensurable to a third *c*, which, by the corollary foregoing, is impossible.

PROPOSITION 5.

187. *As incommensurable quantities cannot be equal one to another, so neither can any multiple, part or parts of one be equal to any multiple, part or parts of the other.*

That two incommensurable quantities, suppose *a* and *b*, cannot be equal is plain; for then either would measure both, and so they would not be incommensurable: and in the next place, that no multiple, part or parts of one can be equal to any multiple, part or parts of the other, I thus demonstrate: Let *c* be any multiple, part or parts of *a*, and *d* of *b*; then if *c* be a multiple of *a*, it is plain that *a* will measure *c*; on the other hand, if *c* be a part of *a*, *c* will measure *a*; and lastly, if *c* be parts of *a*, as $\frac{2}{3}$ of *a*, then it is plain that $\frac{1}{3}$ part of *a* will measure them both; so that in all cases *a* will be commensurable to *c*; and *b* to *d*, for the same reason; therefore *c* and *d* cannot be commensurable; for if they were, since *a* is commensurable to *c*, and *c* to *d*, *a* and *d* would be commensurable, by the last proposition; but if *a* be commensurable to *d*, and *d* to *b*, then *a* and *b* will be commensurable, by the same; but *a* and *b* are not commensurable; therefore neither will *c* and *d*. So far therefore are *c* and *d* from being equal, that they are not so much as commensurable to one another; therefore of two incommensurable quantities, no multiple, part or parts of one can be equal to any multiple, part or parts of the other. *Q. E. D.*

Otherwise more directly thus: *a* is incommensurable to *b*, and *b* is commensurable to *d*; therefore *d* is incommensurable to *a*, by the second corollary to the foregoing proposition: but if *d* be incommensurable to *a*, and *a* be commensurable to *c*, then *d* will be incommensurable to *c*, by the same corollary; therefore &c. *Q. E. D.*

LEMMA.

188. *If of two homogeneous finite quantities* a *and* q, *from the greater* a *be taken more than it's half, and from the remainder more than it's half, and from the last remainder more than it's half,* &c; *I say then that this subtraction may be continued to a remainder less than* q.

For first, since by the supposition *a* and *q* are homogeneous and finite quantities, the quantity *q* cannot be so small but that it may be multiplied till it exceeds *a*, by the third axiom: let then $6q$ be a multiple of *q* that exceeds *a*; then will $\frac{a}{6}$ be less than *q*: let now the half of *a* be ta-

ken

-ken from a, and then there will remain $\frac{1}{2}a$; take away again the half of this, and there will remain $\frac{1}{4}a$; take away again the half of this, and there will remain $\frac{1}{8}a$; but $\frac{1}{8}a$ is less than $\frac{1}{6}a$, and $\frac{1}{6}a$ is less than q; therefore $\frac{1}{8}a$ is less than q; therefore if from a be taken away it's half, and from the remainder it's half, and from the last remainder it's half, and so on, the subtraction may be continued to a remainder less than q; therefore *a fortiori*, if from a be taken away more than it's half, and from the remainder more than it's half, and from the last remainder more than it's half, and so on, the subtraction may be continued to a remainder less than q. *Q. E. D.*

PROPOSITION. 6.

189. *Let* a *and* b *be two homogeneous quantities; and from these two quantities let a continual division be formed in manner following: divide* a *by* b, *and let the remainder be* c; *then divide* b *by* c, *and let the remainder be* d; *then divide* c *by* d, *and let the remainder be* e, &c: *I say then that the series* a, b, c, d, e &c *may be continued to terms less than any assignable quantity as* q.

a.	b.	c.	d.	e.	f.	g.	h.
41.	24.	17.	7.	3.	1.	0.	0.

For in the first division, when a was divided by b, that is, where b was taken away from a as often as possible, the remainder c must be less than b; for if it was not so, b might have been taken out once more, or oftener, contrary to the supposition: since then the quantity taken away was b at least, if not $2b$, $3b$, $4b$ &c, it follows that the quantity taken away must be greater than that which was left; therefore in the first division, where a was divided by b, there was taken away from a more than it's half, and there remained c: in like manner in the third division, where c was divided by d, there was taken away from c more than it's half, and the remainder was e: so again in the fifth division there was taken away from e more than it's half, and the remainder was g, &c; therefore the terms a, c, e, g &c may be continued till they become less than q, by the foregoing lemma; in like manner from the second, fourth and sixth divisions it appears, that the terms b, d, f, h may be continued till they become less than q: put both these considerations together, and then it will appear that the terms a, b, c, d, e, f, g, h may be continued till they become less than q. *Q. E. D.*

PROPOSITION 7.

190. *Supposing all things as in the last proposition; I say then that whatever quantity will measure any two terms of the foregoing series that lie next to one another, the same quantity will measure all the rest both before and after them.*

Let

Let q measure the two contiguous quantities d and e; I ſay then in the firſt place, that q will meaſure all the reſt that follow e: for when d was divided by e there remained the next term f; therefore e meaſures $d-f$: ſince then q meaſures e *ex hypotheſi*, and e meaſures $d-f$, q muſt meaſure $d-f$, by the firſt axiom; but q alſo meaſures d *ex hypotheſi*; therefore q meaſures both the quantities d and $d-f$, whoſe difference is f; therefore q meaſures f by the ſecond axiom: if then becauſe q meaſures d and e, it follows that it muſt neceſſarily meaſure f, ſo becauſe q meaſures e and f, it muſt neceſſarily meaſure g, and ſo on. I ſay in the next place, that if q meaſures d and e, it will meaſure all the terms before d: for when c was divided by d there remained e; therefore d meaſures $c-e$: ſince then q meaſures d *ex hypotheſi*, and d meaſures $c-e$, q meaſures $c-e$; but q meaſures e alſo *ex hypotheſi*; therefore q meaſures both e and $c-e$, whoſe ſum is c; therefore q meaſures c: ſince then upon a ſuppoſition that q meaſures both e and d, it follows that it muſt neceſſarily meaſure c, ſo becauſe q meaſures both d and c, it muſt neceſſarily meaſure b, and ſo on that way. *Q. E. D.*

PROPOSITION 8.

191. *Suppoſing all things as in the two foregoing propoſitions, I ſay that if the original terms* a *and* b *be commenſurable, the ſeries* a, b, c, d, e &c *will not run on* ad infinitum, *but will break off, ſo that the laſt term of the ſeries will be the greateſt common meaſure of* a *and* b; *and* vice verſa, *if the terms* a, b, c, d, e &c *do not run on* ad infinitum, *I ſay then that the original terms* a *and* b *will be commenſurable.*

For firſt, let a and b be commenſurable; I ſay then that the ſeries will not run on *ad infinitum*: for let q be any common meaſure of a and b, then will q meaſure all the ſubſequent terms by the laſt propoſition; but if the terms run on *ad infinitum*, they will come at laſt to be leſs than q by the ſixth propoſition, and ſo q cannot meaſure them all; therefore the ſeries does not run on *ad infinitum*: let then the laſt term be e, and I ſhall then prove in the next place, that e will be greateſt common meaſure of a and b: for as e is the laſt term of the ſeries by the ſuppoſition, it will preciſely meaſure d the laſt but one without any remainder; for ſhould any thing remain, as f, then f would be the next term after e, and ſo e would not be the laſt term of the ſeries, contrary to the ſuppoſition: ſince then e meaſures both d and itſelf, it muſt alſo meaſure a and b by the laſt propoſition; and I ſay further, that it will be the greateſt quantity that can meaſure them both: for ſhould a and b admit of a greater common meaſure than e, ſuppoſe q, then q would meaſure

sure all the subsequent terms, and amongst the rest *e*, by the last proposition, that is, a greater quantity would measure a less, which is absurd; therefore if *e* be the last term of the series, it will be the greatest common measure *a* and *b* will admit of. *Q. E. D.*

I come in the last place to prove, that if the series *a*, *b*, *c*, *d*, *e* &c does not run on *ad infinitum*, the original quantities *a* and *b* will be commensurable: for since the series is supposed not to run on *ad infinitum*, let *e* be the last term; then will *e* measure both *d* and *e* as was proved before; therefore it will also measure *a* and *b* by the last proposition, and therefore *a* and *b* will be commensurable.

COROLLARY I.

Hence if a *and* b *be commensurable, whether they be numbers, or any other homogeneous quantities, it will be easy to find their greatest common measure; to wit, by computing the last term of the series* a, b, c, d, e: thus the greatest common measure of the numbers 42 and 30 is 6, because 6 is the last term of the series 42, 30, 12, 6.

COROLLARY 2.

If the original quantities a *and* b *be incommensurable, the series* a, b, c, d, e &c *will run on* ad infinitum: for if it did not, *a* and *b* would be commensurable, by the latter part of this proposition.

COROLLARY 3.

And vice versa, *if the series* a, b, c, d, e &c *runs on* ad infinitum, *the original quantities* a *and* b *will be incommensurable:* for if they were commensurable, the series would not run on *ad infinitum*, by the former part of this proposition.

COROLLARY 4.

Whatever quantity, as q, *will measure two others* a *and* b, *the same will also measure their greatest common measure* e.

PROPOSITION 9.

A PROBLEM.

192. *It is required, having given three commensurable quantities* a, b *and* c, *to find their their greatest common measure.*

SOLUTION.

First find the greatest common measure of any two of them, suppose of a *and* b, *and call it* d; *then again find the greatest common measure*

of d *and of the third quantity* c, *and call it* e; *then will* e *be the greatest common measure of all the three quantities* a, b *and* c *first proposed.* For first, since *e* measures *d*, and *d* measures both *a* and *b*, *e* will measure both *a* and *b*; but *e* measures *c ex hypothesi*, being supposed the greatest common measure of *d* and *c*; therefore *e* will measure all the three quantities *a*, *b* and *c*. I say in the next place, that *e* is the greatest quantity that can measure them all: for if it be possible, let *f*, a greater quantity than *e*, measure all the three quantities *a*, *b* and *c*; then since *f* measures both *a* and *b*, it will also measure their greatest common measure *d*, by the fourth corollary to the last proposition; but *f* measures *c ex hypothesi*; therefore *f* will measure *e*, the greatest common measure of *d* and *c*, that is, a greater quantity will measure a less, which is absurd; therefore *e* is the greatest common measure of *a*, *b* and *c*. *Q. E. D.*

Of the foregoing rule take the following example in numbers: let it be proposed to find the greatest common measure of the numbers 42, 30 and 15; now the greatest common measure of 42 and 30 is 6, by the first corollary to the eighth proposition; and the greatest common measure of 6 and 15 is 3; therefore the greatest common measure of 42, 30 and 15 is 3; and if the numbers 42, 30 and 15 be all divided by 3, they will be reduced to 14, 10 and 5, three other numbers in the same proportion with the dividends respectively.

COROLLARY.

Whatsoever quantity, as f, *will measure three quantities* a, b *and* c, *the same will measure their greatest common measure* e.

N. B. The following propositions are to prove the existence of incommensurables, and some other affections of them, proper to be known.

PROPOSITION 10.

193. *All equal fractions, when reduced to their least terms, become one and the same fraction.*

Before I enter upon the demonstration of this proposition, it will be proper to take notice by way of lemma, that *If any two numbers, as* 7 *and* 11, *whose greatest common measure is unity, be made to multiply a third quantity, as* d, *of what kind soever, then will that quantity* d *be the greatest common measure of the quantities* 7d *and* 11d; *and* vice versa, *if* d *be the greatest common measure of the quantities* 7d *and* 11d, *then will unity be the greatest common measure of the numbers* 7 *and* 11. For if in finding the greatest common measure of 7 and 11, the divisors be 7, 4, 3, 1, as they actually are, then in finding the greatest common measure of 7*d* and 11*d*, the divisors will be 7*d*, 4*d*, 3*d*, 1*d*; that is, if 1 be the last

diviſor, and conſequently the greateſt common meaſure in the former caſe, d will be the laſt diviſor, and conſequently the greateſt common meaſure in the latter caſe, and *vice verſa.*

This being allowed, let there now be two equal fractions, which when reduced to their leaſt terms, are $\frac{a}{b}$ and $\frac{c}{d}$: I am then to demonſtrate, that the numbers c and d are actually the ſame with the numbers a and b reſpectively.

For firſt it is plain that the numbers a and b can have no common meaſure but unity, any more than the numbers c and d: for if they had, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ would not be in their leaſt terms, but would be further diviſible, contrary to the ſuppoſition.

2dly, The fractions $\frac{a}{b}$ and $\frac{c}{d}$ muſt be equal to each other: for by what methods ſoever theſe reductions to the leaſt terms were made, this is certain, that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ muſt be equal to the fractions firſt propoſed, otherwiſe they would not repreſent thoſe fractions in their leaſt terms, but other fractions: ſince then the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal to the fractions firſt propoſed, and thoſe fractions are equal to each other by the ſuppoſition, it follows that the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are alſo equal.

3dly, Let now the equal fractions $\frac{a}{b}$ and $\frac{c}{d}$ be reduced to the ſame denomination, and the fractions thence ariſing, to wit, $\frac{ad}{bd}$ and $\frac{bc}{bd}$ will ſtill be equal; becauſe neither can this reduction affect their values, by the eighth article of the introduction: ſince then the fractions $\frac{ad}{bd}$ and $\frac{bc}{bd}$ are equal, and have the ſame denominator, they muſt alſo have the ſame numerator, that is, the number bc muſt be the ſame with the number ad; therefore the numbers bc and bd muſt be the ſame with numbers ad and bd reſpectively; therefore the greateſt common meaſure of the former couple will be the ſame with the greateſt common meaſure of the latter; but the greateſt common meaſure of the numbers bc and bd is b, by the lemma, becauſe it has been proved that the greateſt common meaſure of the numbers c and d is unity; and for a like reaſon, the greateſt common meaſure of the numbers ad and bd is d; therefore the

the numbers b and d are the same: since then the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, and have the same denominator, they must likewise have the same numerator, that is, the number c must be the same with the number a; and so the numbers c and d must be actually the same with the numbers a and b respectively. *Q. E. D.*

COROLLARY.

If there be two equal fractions $\frac{a}{b}$ *and* $\frac{c}{d}$, *whereof the former* $\frac{a}{b}$ *is in it's least terms; I say then that* a *the numerator of the former fraction will measure* c *the numerator of the latter, and that* b *the denominator of the former will measure* d *the denominator of the latter as often.* For let e be the greatest common measure of the numbers c and d; then will unity be the greatest common measure of the numbers $\frac{c}{e}$ and $\frac{d}{e}$ by the lemma laid down in the proposition; therefore $\frac{c}{e}$ and $\frac{d}{e}$ will be the least terms of the fraction $\frac{c}{d}$; but the fraction $\frac{c}{d}$ is equal to the fraction $\frac{a}{b}$, which is already in it's least terms by the supposition; and by this proposition the least terms of all equal fractions are the same; therefore the numbers $\frac{c}{e}$ and $\frac{d}{e}$ are the same with the numbers a and b; therefore the numbers c and d are the same with the numbers ea and eb respectively; but a measures ea, and b measures eb as often; therefore a measures c, and b measures d as often. *Q. E. D.*

PROPOSITION II.

194. *If there be three numbers* a, b *and* c, *whereof* a *is prime to* b, *and* b *is a multiple of* c; *I say then that* a *will be prime to* c.

For if a and c be not prime to each other, they must have some common measure greater than unity by the fourth definition; let that be d; then since d measures c, and c measures it's multiple b, d will measure b; but d also measures a, as being a common measure of a and c; therefore d measures both a and b; therefore a and b cannot be prime to each other; therefore if a and c be not prime to each other, neither can a and b; but a and b are prime to each other by the supposition; therefore a and c must also be prime to each other. *Q. E. D.*

SCHO-

SCHOLIUM.

When c is equal to unity, this proposition is self-evident; not so much that the numbers a and c are prime to each other, as that they will have no common measure but unity: should unity be admitted as a number, then every other number would be both prime to it, and a multiple of it; to avoid which confusion, and to establish a distinction betwixt numbers prime and not prime to each other, *Euclid* has excluded unity from his definition of number, chusing rather to consider it as a common denominator of all number: thus the number 3 is to be interpreted three units, &c.

PROPOSITION 12.

195. *If two numbers* a *and* b *be both prime to a third number* c; *I say then that their product* ab *will also be prime to the same third number* c; *or (which amounts to the same thing) that the product* ab *and the number* c *will have no common measure but unity.*

If this be denied, let the product ab and the number c have some common measure greater than unity, and let that be d; then since d measures both the product ab and the number c, it will measure the number c alone, and consequently c will be some multiple of d; therefore we have three numbers a, c and d, whereof a is prime to c, and c is a multiple of d; therefore by the last proposition a is prime to d, and the fraction $\frac{a}{d}$ is in it's least terms: again, since d measures both the product ab and the number c, it will measure ab alone, and consequently ab will be some multiple of d; therefore the fraction $\frac{ab}{d}$ will be equivalent to some whole number; call that whole number e, and then since $\frac{ab}{d}=e$, we shall have $\frac{a}{d}=\frac{e}{b}$: here then we have two equal fractions $\frac{a}{d}$ and $\frac{e}{b}$, whereof the former $\frac{a}{d}$ is in it's least terms; therefore d the denominator of the former fraction will measure b the denominator of the latter, by the corollary to the tenth proposition; but d measures c *ex hypothesi*; therefore d, a number greater than unity, measures both b and c; therefore if the product ab and the number c be not prime to each other, neither can the numbers b and c; but the numbers b and c are prime to each other *ex hypothesi*; therefore the product ab and the number c are also prime to each other. *Q. E. D.*

COROL-

COROLLARY 1.

If two numbers a *and* b *be prime to other two* c *and* d *in ſuch a manner that both the numbers* a *and* b *be prime to each of the numbers* c *and* d; *I ſay then that* ab *the product of the former two will be prime to* cd *the product of the latter two.* For ſince both a and b are prime to c, the product ab will alſo be prime to c by this twelfth propoſition; and for the ſame reaſon the product ab will alſo be prime to the number d: ſince then both the numbers c and d are prime to the number ab, their product cd muſt alſo be prime to ab, and ab to cd. *Q. E. D.*

COROLLARY 2.

If the numbers a and b in the foregoing corollary be ſuppoſed equal to one another, as alſo the numbers c and d; and if moreover a be prime to c, then both a and b will be prime to each of the numbers c and d, and ſo the product ab will be prime to the product cd: but in this caſe, the product ab is the ſame as a^2, and the product cd the ſame as c^2; therefore a^2 will be prime to c^2; which is as much as to ſay, that *If two numbers* a *and* c *be prime to each other, their ſquares* aa *and* cc *will be ſo too.*

COROLLARY 3.

Therefore if any two numbers a *and* c *be the leaſt in their proportion, their ſquares* aa *and* cc *will be the leaſt in their's:* for when either a or c is equal to unity, this is evident of itſelf (unity being incapable of any further diviſion;) and when both a and c are greater than unity, aa and cc will be the leaſt in their proportion by the laſt corollary.

PROPOSITION 13.

196. *If there be three homogeneous quantities* a, b *and* c *in continual proportion, and if the middle term be commenſurable to the extremes; I ſay then that the leaſt numbers in the proportion of the extremes will be both ſquares.*

Since a and b are ſuppoſed commenſurable, let their greateſt common meaſure be contained in a d times, and in b e times; then will d and e be the leaſt numbers in the proportion of a to b, by the tenth propoſition; and ſince a is to b as d is to e, or as dd to de, and ſince b is to c alſo as d to e, or as de to ee, it follows *ex æquo*, that a is to c as dd to ee; but d and e are the leaſt numbers in the proportion of a to b, as hath been already ſhewn; therefore dd and ee are the leaſt numbers in the proportion of a to c, by the third corollary to the laſt propoſition; therefore the leaſt numbers in the proportion of the extremes a and c are both ſquares. *Q. E. D.*

PRO-

PROPOSITION 14.

197. *If there be three homogeneous quantities of any kind* a, b *and* c *in continual proportion, whose extremes* a *and* c *are commensurable to each other; and if the least numbers in the proportion of these extremes be not both squares; I say then that the middle quantity* b *will be incommensurable to both the extremes.*

That the middle quantity cannot be commensurable to one extreme, and incommensurable to the other, is evident; for if we should suppose *a* and *b* for instance, to be incommensurable, and at the same time *b* and *c* to be commensurable, we should then have two incommensurable quantities *a* and *b* both commensurable to the same third quantity *c*, which by the first corollary to the fourth proposition is impossible; therefore the middle quantity must either be commensurable to both the extremes or incommensurable; commensurable it cannot be, for then the least numbers in the proportion of the extremes must be both squares by the last proposition, which contradicts the *hypothesis*; therefore the middle term must be incommensurable to both the extremes. *Q. E. D.*

COROLLARY.

Hence having any line as a *given, it will be easy to find as many others as we please, that shall all be incommensurable to it: for taking any two numbers* d *and* e *that are the least in their proportion, and not both squares, make* c *to* a *as* d *to* e, *and a mean proportional between* a *and* c, *found by the thirteenth of the sixth element, will be incommensurable to both.*

PROPOSITION 15.

198. *If there be any whole number, as* n, *whose square root cannot be expressed by any other whole number; I say then that neither can it be expressed by any fraction whatever.*

For if possible, let the square root of *n* be expressed by a fraction which when reduced to it's least integral terms is $\frac{a}{b}$, that is, let $\frac{a}{b} = \surd n$; then we shall have $\frac{aa}{bb} = \frac{n}{1}$; but the fraction $\frac{aa}{bb}$ is in it's least terms, by the third corollary to the twelfth proposition, because the fraction $\frac{a}{b}$ was so; and the fraction $\frac{n}{1}$ is in it's least terms, because 1 cannot be further reduced; therefore we have two equal fractions $\frac{aa}{bb}$ and $\frac{n}{1}$, both in

in their least terms; therefore by the tenth proposition, these two fractions must not only be equal in their values, but in their terms also, that is, aa must be equal to n, and bb to 1: but aa cannot be equal to n, because a is a whole number by the supposition, and n is supposed to admit of no whole number for it's root; therefore the square root of n cannot possibly be expressed by any fraction whatever. *Q. E. D.*

Otherwise thus: first, n, $\surd n$ and 1 are continual proportionals, because the square of the middle term is equal to the product of the extremes; secondly, the extremes n and 1 are the least in their proportion, because 1 cannot be further reduced; thirdly, and one of the extremes n is no square number by the supposition; therefore the middle term which is $\surd n$ is incommensurate to 1; therefore $\surd n$ cannot possibly be expressed by any fraction or mixt number whatever, because these are all commensurate to 1 by the third corollary to the definitions. *Q. E. D.*

SCHOLIUM.

From this proposition it appears, that two surd numbers may be both incommensurable to unity, and yet both commensurable to one another: for $\surd 2$ and $\surd 8$ (by this proposition) are both incommensurable to unity, and yet they will both be commensurable to one another; for since 2 is to 8 as 1 is to 4, $\surd 2$ will be to $\surd 8$ as 1 to 2.

PROPOSITION 16.

199. *If there be two numbers* a *and* b *the least in their proportion, and such whose square roots are commensurable one to the other; I say that the numbers* a *and* b *must be both squares.*

For since $\surd a$ is supposed commensurable to $\surd b$, let their greatest common measure be contained in $\surd a$, d times, and in $\surd b$, e times; then will d and e be the least numbers in the proportion of $\surd a$ to $\surd b$ by the tenth proposition; and since $\surd a$ is to $\surd b$ as d to e, we shall have a to b as dd to ee; whence $\frac{a}{b} = \frac{dd}{ee}$ by the fourth axiom: but the fraction $\frac{a}{b}$ is in it's least terms by the supposition, and so also is the fraction $\frac{dd}{ee}$, because the fraction $\frac{d}{e}$ was; therefore $a = dd$, and $b = ee$; therefore a and b are both squares. *Q. E. D.*

COROLLARY.

Therefore if a *and* b *be the least in their proportion, and not both squares, as* 1 *and* 2, 2 *and* 3, &c, *their square roots will be incommensurable to one another.*

PROPOSITION 17.

200. *If two incommensurable quantities as 2 and $\sqrt{3}$ be put together, so as to constitute a third quantity $\overline{2+\sqrt{3}}$; I say then that the quantity so constituted shall be incommensurable to both the constituent parts.*

Thus I shall prove that $\overline{2+\sqrt{3}}$ is incommensurable to $\sqrt{3}$.

For should it be otherwise, both $\overline{2+\sqrt{3}}$ and $\sqrt{3}$ would have some common measure; let that be m; and then since m measures both $\overline{2+\sqrt{3}}$ and $\sqrt{3}$, it will also measure their difference 2, and so m will measure both 2 and $\sqrt{3}$; but 2 and $\sqrt{3}$ are incommensurable by the supposition; therefore $\overline{2+\sqrt{3}}$ must not be commensurable to $\sqrt{3}$. *Q. E. D.* And the same demonstration may be applied, *mutatis mutandis*, to prove that $\overline{2+\sqrt{3}}$ is incommensurable to 2.

PROPOSITION 18.

201. *The side and diameter of a square are incommensurable.*

This follows from the fifteenth proposition; for by the fortyseventh of the first element, if the side of a square be called 1, it's diameter will be $\sqrt{2}$; but 1 and $\sqrt{2}$ are incommensurable by the fifteenth proposition; therefore the side and diameter of a square are incommensurable. *Q. E. D.*

But this proposition may also be demonstrated independently of the foregoing propositions by the help of the following lemma.

LEMMA.

If a square number be even, not only it's root, but it's half will be so too.

For an odd root as $2a+1$ produces an odd square, as $4aa+4a+1$; whereas an even root, as $2a$, produces not only an even square, as $4aa$, but it's half $2aa$ is likewise even.

This premised, if the side and diameter of any square be not incommensurable, they must have a common measure: let their greatest common measure be contained in the side a times, and in the diameter b times; then will a and b be the least numbers expressing the proportion of the side of a square to it's diameter: and since the side is to the diameter as a to b, the square of the side will be to the square of the diameter as aa to bb: but the square of the side is half the square of the diameter, by the fortyseventh of the first element; therefore aa is half of bb; therefore aa and bb are both even numbers, bb for having an exact half, and aa for being that half, according to the lemma; therefore by the

ſame lemma, the roots of theſe even ſquares, to wit, a and b muſt both be even numbers; for if either of them was odd, it's ſquare muſt be ſo too: but now as theſe numbers are the leaſt in their proportion, one of them at leaſt muſt be an odd number, or the proportion would ſtill be reducible to lower terms; therefore if the ſide and diameter of a ſquare be not incommenſurable, it will be poſſible for one and the ſame number to be both even and odd; but this is impoſſible; therefore they are incommenſurable. *Q. E. D.*

Obſervations upon the whole.

202. *Scholium* 1. Thus we ſee how much more ſubtil the parts of continued quantities are than thoſe of numbers; for though the root of 2 cannot poſſibly be expreſſed by any whole number or fraction, yet we ſee that if the ſide of any ſquare be called 1, whether it be 1 yard, 1 foot, 1 inch, or whatever it is, the diameter will be the root of 2 upon the ſame ſcale; nay it will be very eaſy to conſtruct a ſcale exhibiting the ſquare-roots of as many numbers as we pleaſe in their natural order from unity: but of this more in another place; ſee art. 307, ſcholium.

By this propoſition we are alſo given to underſtand, that the areas of two ſquares may be commenſurable, and at the ſame time the ſides be incommenſurable; for the area of any ſquare is to the area of another ſquare upon the diameter of the former as 1 to 2; and yet according to the laſt propoſition, the ſides of theſe two ſquares are incommenſurable.

Scholium 2. In the corollary to the third propoſition it is demonſtrated, that if incommenſurability can be proved to belong to any one ſort of continued quantity whatever, it muſt equally belong to any other; but in the laſt propoſition and in the corollary to the fourteenth it is demonſtrated, that lines may be incommenſurable; therefore all other continued quantities may. I ſhall only produce one inſtance of incommenſurability in any other ſort of quantity, and that ſhall be in time.

Monſieur *Huygens*, in his admirable treatiſe of the motion of pendulums, has demonſtrated that the lengths of all pendulums, reckoning from the center of ſuſpenſion to the center of oſcillation, are as the ſquares of the times wherein they perform ſimilar oſcillations reſpectively: as if the length of one pendulum be three philoſophical feet, and that of anotwo ſuch feet, the ſquare of the time of an oſcillation of the former pendulum will be to the ſquare of the time of a ſimilar oſcillation of the latter as 3 to 2; whence the times themſelves will be as $\sqrt{3}$ to $\sqrt{2}$, and conſequently may be repreſented by theſe two ſurds: let us now ſuppoſe what is pretty near the caſe, to wit, that $\sqrt{3}$ is to $\sqrt{2}$ as 49 to 40; then by multiplying extremes and means we ſhall have $\sqrt{3} \times 40 = \sqrt{2} \times 49$,

$\times 49$, which is as much as to ſay that forty oſcillations of the longer pendulum are performed in the ſame time as fortynine of the ſhorter: now if this was exactly true, and if two clocks furniſhed with theſe pendulums ſhould happen at any particular inſtant of time to beat together, after 40 beats cauſed by the longer pendulum, or 49 by the ſhorter, this coincidence would return again, and ſo on *ad infinitum*: this is upon a ſuppoſition that $\sqrt{3}$ is to $\sqrt{2}$ as 49 to 40, and conſequently that $\sqrt{3} \times 40 = \sqrt{2} \times 49$; but if we ſuppoſe theſe two ſurds, and conſequently the times they repreſent, to be (as they truly are) incommenſurable, no multiple of one can be equal to any multiple of the other by the fifth propoſition; and then it will follow that if at any particular inſtant of time theſe two clocks ſhould happen to beat together, there can never happen ſuch another inſtant, though their motion ſhould continue to all eternity, provided the force of the wheels have no other influence upon theſe motions than to prevent retardations ariſing from other cauſes: they will frequently beat ſo cloſe together that it will not be in the power of the ear to diſtinguiſh the difference; but another true mathematical coincidence can never happen after the firſt.

Scholium 3. Before I take my leave of this ſubject, I ſhall only further obſerve, that this doctrine of incommenſurables quite overturns the *hypotheſis* of indiviſibles: for were there any ſuch things as indiviſibles, or leaſt parts of magnitude, theſe would meaſure all the reſt, ſince they could leave nothing leſs than themſelves, and ſo there would be no ſuch things as incommenſurables, the exiſtence of which we have already demonſtrated beyond all contradiction.

PROBLEM I.

203. *Let* a *and* b *be two incommenſurable quantities,* a *a greater and* b *a leſs; and ſuppoſing a continual diviſion to be made from* a *and* b, *according to the method for finding the greateſt common meaſure, let the quotients thence ariſing be certain numbers always returning in the ſame order* ad infinitum: *It is required to determine the value of the fraction* $\frac{a}{b}$, *or (which is the ſame thing) to determine the proportion of* a *to* b *without any approximation, admitting ſurd numbers into the expreſſion.*

N. B. To determine this proportion by an approximation at pleaſure, ſee ſcholium 1 to art. 179.

SOLUTION.

Let the returning quotients be 1, 2, 3. 1, 2, 3. 1, 2, 3. *&c ad infinitum*, and let the remainders of the firſt, ſecond and third diviſions be

c, d and e reſpectively; then will the quotient of a divided by b be 1, that of b divided by c, 2, that of c divided by d, 3; after which the quotient of d divided by e will again be 1, and ſo on: therefore a continual diviſion begun from d and e and carried on *ad infinitum*, will be attended with the ſame quotients both in quantity, order and number as a continual diviſion begun from a and b; therefore d will have the ſame proportion to e as a to b, by art. 180. This being allowed, let a ſeries of equations for a and b be computed as in the 175th article, and they will be as follows:

$$1a - 0b = +a.$$
$$0a - 1b = -b.$$
$$1a - 1b = +c.$$
$$2a - 3b = -d.$$
$$7a - 10b = +e. \text{ \&c.}$$

Here then $d = 3b - 2a$, and $e = 7a - 10b$; but it has been proved above that d is to e as a to b; therefore $3b - 2a$ is to $7a - 10b$ as a to b; therefore by multiplying extremes and means, we have $3bb - 2ab = 7aa - 10ab$, or $7aa - 8ab = 3bb$: here then we have but one equation $7aa - 8ab = 3bb$ for determining two unknown quantities a and b, and therefore are at liberty to ſubſtitute what we pleaſe for one of them: let us then ſubſtitute ſuch a quantity for b as will render the equation more ſimple thus; let $b = 7$, becauſe 7 is the coefficient of aa in the equation; and then dividing the whole equation by 7, or it's equal b, we ſhall have $aa - 8a = 3b$, or $aa - 8a = 21$; this is a quadratic equation, and by compleating the ſquare we ſhall have $aa - 8a + 16 = 37$; whence by extracting the ſquare root we have $a - 4 = \sqrt{37}$, and $a = 4 + \sqrt{37}$; therefore $\frac{a}{b} = \frac{4 + \sqrt{37}}{7}$, or (which is the ſame thing) a is to b as $4 + \sqrt{37}$ is to 7. *Q. E. I.*

N. B. 1*ſt*. Though the number 37, like all others, has two roots, yet I chuſe the affirmative one, becauſe $4 - \sqrt{37}$ is a negative quantity.

2*dly*. If you would expreſs the numerator of the fraction $\frac{a}{b}$ by a whole number, and the denominator by a mixt ſurd, in the foregoing equation $7aa - 8ab = 3bb$ you muſt make $a = 3$ the coefficient of bb, and then dividing the whole equation by a or 3, you will have $7a - 8b = bb$, that is, $bb + 8b = 21$, which being reſolved gives $b = \sqrt{37} - 4$; whence $\frac{a}{b} = \frac{3}{\sqrt{37} - 4}$, or a is to b as 3 is to $\sqrt{37} - 4$.

3*dly*. To confirm theſe two proportions, to wit, that a is to b as $\sqrt{37} + 4$ is to 7, and alſo that a is to b as 3 is to $\sqrt{37} - 4$, we are to

to take notice that the ſquare root of 37 is 6 .08276253 nearly, and conſequently that $\sqrt{37}+4$ is to 7 as 10 .08276253 is to 7, or as 1008276253 is to 700000000. Now if a continual diviſion be made from the numbers 1008276253 and 700000000, the quotients will be found to be 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, the law of the ſeries now breaking off, becauſe the ſquare root of 37 here given was not exact: on the other hand, that a is to b alſo as 3 to $\sqrt{37}-4$, that is, as 300000000 is to 208276253, will be evident by forming a continual diviſion from the numbers 300000000 and 208276253, where the quotients will be found to be 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, &c.

4thly. From the ſolution of the foregoing problem it appears, that the ſquare root of 37 cannot poſſibly be expreſſed by any whole number, mixt number, or fraction of what kind ſoever; for if it could, then $4+\sqrt{37}$ might be expreſſed ſo too, and conſequently would not be incommenſurable to the whole number 7, as by the infinity of quotients we find it is: and the ſame may be obſerved of all other ſurds found by the method of this ſolution.

5thly. If we would form a theorem for a general ſolution of the foregoing problem, we muſt look back a little upon the particular one; and conſidering it more attentively, we ſhall find a method of ſolving it in general thus: after a ſeries of equations was formed according to art. 175 by the help of one circulation of the quotients 1, 2, 3, we found the two laſt equations to be $2a-3b=-d$, and $7a-10b=+e$; whence I infer, that if (according to art. 179) a ſeries of primary fractions be computed by the help of the quotients 1, 2, 3, the two laſt fractions in the ſeries will be $\frac{3}{2}$ and $\frac{10}{7}$: let us now examine the laſt equation from which the quantity a was found, to wit, $aa-8a=21$ or $3b$, and upon reflection we ſhall find that 8 the coefficient of $-a$ in the ſecond term was found by ſubtracting 2 from 10, that is, by ſubtracting the denominator of the laſt fraction but one from the numerator of the laſt: we ſhall find alſo that 7 the value of b came out from 7 the denominator of the laſt fraction: laſtly we ſhall find that 3 the coefficient of b in the term $3b$, or of 7 in the number 21 came from 3 the numerator of the laſt fraction but one, and conſequently that $3b$ or 21 was the product of 3 the numerator of the laſt fraction but one multiplied into b or 7 the denominator of the laſt: all which conſiderations put together furniſh us with the following general ſolution.

By the help of the firſt circulation of quotients compute (as in article one-hundred ſeventynine) a ſeries of primary fractions; which done, multiply the numerator of the laſt fraction but one into the denominator of the laſt, and call the product p; *ſubtract the denominator of the laſt fraction but one from the*

the numerator of the last, and call the difference d; *then you will have* b *equal to the denominator of the last fraction, and* aa—da=p.

EXAMPLE 1.

Let the recurring quotients be 4, 1, 1, 1. 4, 1, 1, 1. *&c ad infinitum*: now a series of primary fractions computed by the means of the quotients 4, 1, 1, 1 is $\frac{0}{1}$, $\frac{1}{0}$, $\frac{4}{1}$, $\frac{5}{1}$, $\frac{9}{2}$, $\frac{14}{3}$; and 9 the numerator of the last fraction but one multiplied into 3 the denominator of the last gives $27=p$; and moreover 2 the denominator of the last fraction but one subtracted from 14 the numerator of the last leaves $12=d$: make therefore $b=3$ the denominator of the last fraction, and you will have $aa-12a=27$; which equation resolved gives $a=6+\sqrt{63}$; therefore $\frac{a}{b}=\frac{6+\sqrt{63}}{3}$: but this fraction may be reduced lower; for $\frac{6}{3}=\frac{2}{1}$, and $\frac{\sqrt{63}}{3}=\frac{\sqrt{63}}{\sqrt{9}}=\sqrt{\frac{63}{9}}=\sqrt{\frac{7}{1}}$; therefore $\frac{a}{b}=\frac{2+\sqrt{7}}{1}$.

EXAMPLE 2.

Let the quotients be q, q, q, q, q *ad infinitum*; then a series of fractions computed by the quotient q will be $\frac{0}{1}$, $\frac{1}{0}$, $\frac{q}{1}$, where the two last fractions are $\frac{1}{0}$ and $\frac{q}{1}$: now 1 the numerator of the last fraction but one multiplied into 1 the denominator of the last gives $1=p$; and 0 the denominator of the last fraction but one subtracted from q the numerator of the last leaves $q=d$; make therefore $b=1$ the denominator of the last fraction, and you will have $aa-qa=1$: let $q=2$, and the equation will be $aa-2a=1$; whence $a=1+\sqrt{2}$, and $\frac{a}{b}=\frac{1+\sqrt{2}}{1}$; therefore if a continual division be made from the quantities $1+\sqrt{2}$ and 1, the quotients will be 2, 2, 2, 2, 2 *ad infinitum*; but if $\sqrt{2}$ be divided by 1, the divisor and the remainder will be the same as when $1+\sqrt{2}$ was divided by 1; therefore if $\sqrt{2}$ be divided by 1, all the quotients after the first will be the same as when $1+\sqrt{2}$ was divided by 1; but the first quotient in the former case will be 1, and not 2 as in the latter; therefore if a continual division be formed from the quantities $\sqrt{2}$ and 1, the first quotient will be 1, and the rest will be 2, 2, 2, 2, 2 *ad infinitum*: now every one that knows any thing of Geometry, knows that

that the diagonal is to the ſide of any ſquare as $\surd 2$ to 1; whence it follows from the fourth obſervation of this article, that the diagonal and ſide of a ſquare are incommenſurable, and that if a continual diviſion be formed from them, the firſt quotient will be 1, and the reſt will be 2, 2, 2, 2, 2 *ad infinitum*. Again, let $q=1$, and we ſhall have $b=1$ as before, and $aa-a=1$: let the greater ſegment of a line cut in extreme and mean proportion be to the leſs as a to 1; then ſince (according to the nature of this ſection) the whole line is to the greater ſegment as the greater ſegment is to the leſs, we have this proportion, $a+1$ is to a as a is to 1; and this equation, $aa=a+1$, and $aa-a=1$: ſince then we fall into the ſame equation, whether we ſuppoſe $q=1$, or ſuppoſe a to be to 1 or b as the greater ſegment of a line divided in extreme and mean proportion is to the leſs, it follows that a is to b in that proportion; and conſequently that if a continual diviſion be formed from thoſe ſegments, the quotients will be 1, 1, 1, 1, 1 *ad infinitum*.

EXAMPLE 3.

Let the circulating quotients be $q, r.\ q, r.\ q, r.$ &c *ad infinitum*: here a ſeries of primary fractions computed from the quotients q and r will be $\frac{0}{1}, \frac{1}{0}, \frac{q}{1}, \frac{qr+1}{r}$; where q the numerator of the laſt fraction but one multiplied into r the denominator of the laſt gives $qr=p$; and 1 the denominator of the laſt fraction but one ſubtracted from $qr+1$ the numerator of the laſt leaves alſo $qr=d$; make therefore $b=r$ the denominator of the laſt fraction, and a will be determined by this equation, to wit, $aa-qra=qr$: if $q=2$, and $r=1$, we ſhall have a to b as $1+\surd 3$ is to 1; therefore in a continual diviſion begun from $1+\surd 3$ and 1, the quotients will be 2, 1. 2, 1. 2, 1. &c *ad infinitum*; but in a continual diviſion begun from $\surd 3$ and 1, the firſt quotient will be 1, and the reſt will be 1, 2. 1, 2. 1, 2. &c *ad infinitum*. If $q=1$, and $r=3$, a will be to b as $\frac{3+\surd 21}{2}$ is to 3, or as $3+\surd 21$ to 6.

If ſome of the firſt quotients are intended to be irregular, and the reſt to return in a circle *ad infinitum*, proceed as follows: *Let it be required that* n *and* p *ſhall be the two firſt quotients, and that the reſt ſhall be* q, r, s. q, r, s. q, r, s. &c ad infinitum: *firſt then find by the foregoing problem two quantities* a *and* b *that ſhall have the regular quotients above deſcribed* ad infinitum *without any others; then make* $pa+b=B$, *and* $nB+a=A$, *and the two quantities* A *and* B *will have the quotients required.* For ſince $A=nB+a$, it follows that if A be divided by B, the quotient will be n, and the remainder will be a: in like manner, ſince $B=pa+b$, it follows

follows that if B be divided by a, the quotient will be p, and the remainder will be b: the next diviſion is that of a by b, and ſo on; and therefore if this diviſion be continued *ad infinitum*, the reſt of the quotients will be $q, r, s.\ q, r, s.\ q, r, s.$ *&c ad infinitum* by the ſuppoſition.

PROBLEM 2.

204. *I owe a friend a ſhilling, or ſome number of ſhillings, and we have nothing but guineas and louiſd'ors about us; the guineas being valued at twentyone ſhillings apiece, and the louiſd'ors at ſeventeen: The queſtion is, How muſt I acquit myſelf of this debt?*

SOLUTION.

Since by the ſuppoſition this debt cannot otherwiſe be diſcharged than either by my paying guineas and receiving louiſd'ors, or elſe paying louiſd'ors and receiving guineas, or perhaps paying both guineas and louiſd'ors, it follows that this debt muſt be either the ſum or the difference of a certain number of guineas and a certain number of louiſd'ors, and conſequently that it muſt either be one ſhilling, which is the greateſt common meaſure of a guinea and a louiſd'or, or elſe ſome number of ſhillings without a fraction; for nothing is more evident than that whatever meaſures any two quantities, muſt meaſure not only their multiples, but alſo the ſums and differences of thoſe multiples, as hath been before obſerved: and this is all the limitation this problem is ſubject to. This being then laid down, I make $a=21$, and $b=17$, and from theſe two values of a and b I derive a ſet of equations as in art. 175, *viz.*

$$1a - 0b = +21.$$
$$0a - 1b = -17.$$
$$a - b = +4.$$
$$4a - 5b = -1.$$
$$13a - 16b = +1.$$

Of theſe equations the two laſt will ſolve the problem as follows.

CASE 1.

Let the debt be one ſhilling: then if I would make my payment in guineas and receive louiſd'ors, of the two laſt equations I make choice of that whoſe abſolute term is affirmative, to wit, $13a - 16b = +1$, whereby it manifeſtly appears that if I pay my friend thirteen guineas, and he returns back again to me ſixteen louiſd'ors, the debt will be diſcharged; for paying thirteen guineas and receiving ſixteen louiſd'ors is equivalent to paying thirteen guineas *minus* ſixteen louiſd'ors, which ac-

cording to the equation above quoted amounts to juſt one ſhilling: on the other hand, if I would make my payment in louiſd'ors and receive guineas, of the two laſt equations I muſt have recourſe to that whoſe abſolute term is negative, to wit, $4a - 5b = -1$, or $5b - 4a = +1$, whereby it appears that if I pay five louiſd'ors and receive back four guineas, I ſhall have diſcharged the debt: and as theſe equations $4a - 5b = -1$, and $13a - 16b = +1$ are the ſimpleſt of their kind by art. 177, the ſolutions deduced from them muſt be ſo too.

Now to find other ſolutions of this problem; ſince a is to b as 21 to 17, we have (by multiplying extremes and means) $17a = 21b$, or $17a - 21b = 0$; and as the fraction $\frac{21}{17}$ is in it's leaſt terms, the eqnation $17a - 21b = 0$ is the ſimpleſt of it's kind: add now the equation $17a - 21b = 0$ to the equation $13a - 16b = 1$ as often as you pleaſe by a continual addition, and alſo the equation $21b - 17a = 0$ to the equation $5b - 4a = 1$ in like manner, and you will have an infinite number of ſolutions to this problem, whether the payment is to be made in guineas and the balance in louiſd'ors, or the payment in louiſd'ors and the balance in guineas.

CASE 2.

Let now the debt be five ſhillings: then if I would pay guineas and receive louiſd'ors, I multiply the equation $13a - 16b = 1$ by 5 the number expreſſing the debt, and ſo have $65a - 80b = 5$: now to find the ſimpleſt equation of this kind, that is, whoſe abſolute term is $+5$, I ſubtract, as often as I can, the equation $17a - 21b = 0$ from the equation $65a - 80b = 5$; and to do this the ſhorteſt way, I divide $65a$ by $17a$, that is, 65 by 17, and find the quotient to be 3 with a remainder; whence I infer that I can take the equation $17a - 21b = 0$ three times out of the equation $65a - 80b = 5$, and yet leave the coefficient of a affirmative: now $17a - 21b = 0$ being multiplied by 3 gives $51a - 63b = 0$, and this ſubtracted from $65a - 80b = 5$ gives $14a - 17b = 5$; therefore the ſimpleſt way of diſcharging the debt by making payment in guineas, is to pay 14 guineas and to receive 17 louiſd'ors: and if from this laſt equation $14a - 17b = 5$ again be ſubtracted $17a - 21b = 0$, there will remain $4b - 3a = 5$, which gives the ſimpleſt way of making the payment in louiſd'ors.

CASE 3.

Laſtly let the debt be 100 pounds or 2000 ſhillings: then to ſolve this caſe, multiply the equation $13a - 16b = 1$ by 2000, and you will have $26000a - 32000b = 2000$: for a more ſimple equation, divide

26000

26000 by 17, to wit, the coefficient of a in the equation $26000a - 32000b = 2000$ by the coefficient of a in the equation $17a - 21b = 0$, and the quotient will be 1529 with a remainder: ſubtract therefore 1529 times the equation $17a - 21b = 0$, that is, the equation $25993a - 32109b = 0$ from the equation $26000a - 32000b = 2000$, and you will have the equation $7a + 109b = 2000$; which ſhews that if I pay my friend 7 guineas and 109 louiſd'ors, I ſhall diſcharge the debt of one hundred pounds without receiving any thing back by way of exchange.

This then is one way of making up one hundred pounds out of guineas and louiſd'ors; and as many other ways may be found as the equation $17a - 21b = 0$ can be added to the equation $7a + 109b = 2000$, without abſolutely deſtroying the coefficient of b, or reducing it to a negative: to know therefore how often this can be done, divide 109 the coefficient of b in the equation $7a + 109b = 2000$ by 21 the coefficient of $-b$ in the equation $17a - 21b = 0$, and the quotient will be 5 with a remainder; therefore there are five other ways of making up one hundred pounds out of guineas and louiſd'ors beſides that already found, and conſequently ſix ways in all, to wit,

7 guineas and 109 louiſd'ors,
24 guineas and 88 louiſd'ors,
41 guineas and 67 louiſd'ors,
58 guineas and 46 louiſd'ors,
75 guineas and 25 louiſd'ors, and
92 guineas and 4 louiſd'ors.

The firſt of theſe ways ſolves the problem with the feweſt guineas, and laſt with the feweſt louiſd'ors poſſible, and therefore may be called the extreme ſolutions of the problem: but if the extreme ſolutions be required without the intermediates, then to one of the extreme equations already found, to wit, $7a + 109b = 2000$ add 5 times the equation $17a - 21b = 0$, to wit, the equation $85a - 105b = 0$, becauſe there were five other ſolutions found, and you will have $92a + 4b = 2000$, which is the other extreme equation.

Beſides the ſix ways of diſcharging the debt without interchanging, there are infinite other ways of doing it by way of interchange, as will appear by a continual addition and ſubtraction of the equation $17a - 21b = 0$. *Q. E. I.*

A DEFINITION.

205. *Let* q, r, s, t *be any numbers given, and let* a, b, c, d, e, f *be a ſeries whereof the two firſt terms* a *and* b *are known, and the reſt are obtained in the following manner: make* $qb + a = c$, $rc + b = d$, $sd + c = e$, $te + d = f$; *then may the ſeries* a, b, c, d, e, f *be ſaid to be formed or ge-*

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nerated from it's two first terms a *and* b *by the numbers* q, r, s, t. Thus a series generated from 1 and 0 by the numbers 2, 3, 4, 5, 6 will be 1, 0, 1, 3, 13, 68, 421; and a series generated from 0 and 1 by the numbers 3, 4, 5, 6 will be 0, 1, 3, 13, 68, 421: whence it appears, that *If a series be formed from* 1 *and* 0 *by any set of numbers* 2, 3, 4, 5, 6; *and then setting aside the first number* 2, *another series be formed from* 0 *and* 1 *by the remaining numbers* 3, 4, 5, 6, *this latter series will be the same with the former, after the first term* 1 *is taken away.*

PROBLEM 3.

206. *It is required to find as many numbers as we please of this character, to wit, that if any one of them be divided by two given divisors,* a *a greater, and* b *a less, they shall have two given remainders* d *and* e *respectively.*

SOLUTION.

Let g be any one of the numbers sought, and let l be the greatest common measure of the two given divisors a and b; then since l measures a, and a measures $\overline{g-d}$ by the supposition, it follows that l must measure $\overline{g-d}$; in like manner, since l measures b, and b measures $\overline{g-e}$, l must measure $\overline{g-e}$; therefore l measures both the numbers $\overline{g-d}$ and $\overline{g-e}$; therefore if the former number be subtracted from the latter, l must measure the remainder $\overline{d-e}$; therefore if this problem be possible, $\frac{d-e}{l}$ must be reducible to a whole number, either affirmative or negative as d happens to be greater or less than e. This premised, let now the two values of a and b be particularly assigned; as for example, let $a=105$ and $b=40$, and let a series of equations be formed from these two values of a and b according to art. 175, as follows:

Equation 1, $1a-0b=+105$.
2, $0a-1b=-40$.
3, $1a-2b=+25$.
4, $1a-3b=-15$.
5, $2a-5b=+10$.
6, $3a-8b=-5$.
7, $5a-13b=+5$.

From the two last of these equations it appears that 5 is the greatest common measure of a and b, (see art. 175;) and consequently that $\frac{d-e}{5}$ must be a whole number in all possible cases of this problem. Take now the

the laſt equation of the ſeries whoſe abſolute term is negative, as $3a-8b=-5$, and you will have $3a+5=8b$; multiply the terms of this laſt equation into the number $\frac{d-e}{5}$, and you will have $3a\times\frac{d-e}{5}+5\times\frac{d-e}{5}=8b\times\frac{d-e}{5}$, that is, $3a\times\frac{d-e}{5}+d-e=8b\times\frac{d-e}{5}$; tranſpoſe $-e$, and you will have $3a\times\frac{d-e}{5}+d=8b\times\frac{d-e}{5}+e$: but the number $3a\times\frac{d-e}{5}$ being divided by a, quotes $3\times\frac{d-e}{5}$, and there remains nothing; therefore the number $3a\times\frac{d-e}{5}+d$ being divided by a, there will remain d; in like manner if the number $8b\times\frac{d-e}{5}+e$ be divided by b, there will remain e: ſince then $3a\times\frac{d-e}{5}+d$ is equal to $8b\times\frac{d-e}{5}+e$, the ſame number $3a\times\frac{d-e}{5}+d$, or $\frac{3a}{5}\times\overline{d-e}+d$ will anſwer both the conditions of the problem; but $a=105$ *ex hypotheſi*; therefore $\frac{3a}{5}\times\overline{d-e}+d=63\times\overline{d-e}+d$; therefore the number $63\times\overline{d-e}+d$ will anſwer both the conditions.

This one ſolution being obtained, innumerable others may be had by a continual addition and ſubtraction of 840 the leaſt common multiple of a and b, which how to find, ſee art. 171: for as this number being ſeverally divided by 105 and 40 will have no remainder, it follows that if this common multiple, or any number of them be added to or ſubtracted from $63\times\overline{d-e}+d$, the remainders, when the diviſion is made by a and b, will be entirely the ſame as before that addition or ſubtraction, and therefore the remainders will ſtill be d and e, and this will give us an infinity of numbers with the properties above deſcribed. *Q. E. I.*

Thus have we a general theorem computed for the diviſors 105 and 40, be the remainders d and e what they will, provided $\frac{d-e}{5}$ be a whole number: as for example, let it be required to find a number which being ſeverally divided by 105 and 40, will have 39 and 9 for the reſpective remainders: here $d=39$, $e=9$, $d-e=30$, $63\times\overline{d-e}=1890$, $63\times\overline{d-e}+d=1929$; therefore 1929 is ſuch a number as will anſwer the conditions of the queſtion: but if you would have the leaſt number

number poſſible that will do the ſame thing, then caſt away the number 840 (the leaſt common multiple of 105 and 40) as often as poſſible, from 1929, that is, divide 1929 by 840, and the remainder 249 will be the leaſt number which being divided by 105 and 40 will have 39 and 9 for their reſpective remainders. Again, let the remainders, inſtead of being 39 and 9, be 9 and 39 for a and b reſpectively; then will $d-e$ be -30, and $63\times\overline{d-e}=-1890$, and $63\times\overline{d-e}+d=-1890+9=-1881$; divide -1881 by 840, and the remainder is -201, which is the leaſt negative number of this kind; but we want the leaſt affirmative number; and therefore if to the negative number already found, to wit, -201 be added the leaſt common multiple 840, you will have 639 the leaſt affirmative number which being divided by 105 and 40 will have 9 and 39 reſpectively remaining.

If any one be deſirous of a ſynthetical demonſtration to prove that the number $63\times\overline{d-e}+d$, or $64d-63e$ being ſeverally divided by 105 and 40 will have d and e remaining reſpectively, he is to take notice, that if the diviſors 105 and 40 had been prime to each other, this would eaſily have appeared by a ſimple diviſion of the number $64d-63e$ by 105 and 40; but as it is otherwiſe here, ſome further conſiderations are neceſſary, which are as follows. Since, when $64d-63e$ is divided by 105, the laſt remainder is to be d, therefore I exterminate the other quantity $-63e$ thus: the greateſt common meaſure of 105 and 40 is 5; therefore $\frac{d-e}{5}$ is a whole number, by what has been proved already; now $63\times\overline{d-e}$ is equal to $5\times63\times\frac{d-e}{5}$, or $315\times\frac{d-e}{5}$, both factors whereof are whole numbers; and $315\times\frac{d-e}{5}$ is diviſible by 105 without any remainder, becauſe 315 is ſo; therefore $63d-63e$ is diviſible by 105 without any remainder; therefore if $63d-63e$ be added to or ſubtracted from any number whatever, and the ſum or the remainder be divided by 105, the remainder of this diviſion will be the ſame as if ſuch addition or ſubtraction had never been made: ſubtract then $63d-63e$ from $64d-63e$, and there remains d; therefore $64d-63e$ being divided by 105 will have d for a remainder. In the next place I divide $64d-63e$ by 40, and there remains $24d-23e$; but if the theorem be true, the number e ought to be the laſt remainder; to try which, I expunge the other quantity $24d$ out of the remainder $24d-23e$ thus: $24\times\overline{d-e}$, or $120\times\frac{d-e}{5}$ is a number diviſible by 40 with-

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out a remainder; therefore $24d-24e$ is divisible by 40; subtract then $24d-24e$ from $24d-23e$, and there remains e; therefore $64d-63e$ being divided by 40 will have e for a remainder. *Q. E. D.*

We come now to form a general theorem for any two divisors a and b whatsoever, where the remainders are possible; and therefore must look back upon the foregoing solution, where it was found that the number $\frac{3a}{5}\times\overline{d-e}+d$ would answer the conditions of the question. Now any one who observes how this solution was obtained, will easily perceive that in this expression $\frac{3a}{5}\times\overline{d-e}+d$, the number 3 was the coefficient of a in the last equation of the foregoing series whose absolute term is negative, and that the number 5 was the greatest common measure of the two given divisors a and b: make then $r=3$, $l=5$ as before, and the expression $\frac{3a}{5}\times\overline{d-e}+d$ will now be changed into this, $\frac{ra}{l}\times\overline{d-e}+d$: but the greatest difficulty still remains behind, which is to trace out the number r by itself without being obliged to compute all the equations in the foregoing series, that is, to find out the coefficient of a in the last equation of those whose absolute terms are negative, or (which is the same thing) to find out the coefficient of a in the last equation that possesses an even place in the series; for by a bare inspection of these serieses, every one may see that all the even places are taken up by equations whose absolute terms are negative. Now whoever considers with any degree of attention the nature and *genesis* of these equations, cannot but see that the coefficients of a, from the first equation to the last, are a series of numbers computed from 1 and 0 by the quotients of a continual division from the divisors a and b; and from the last article it appears that such a series is always equivalent to a series begun from 0 and 1, and computed by the same quotients except the first; therefore in finding the number r, the first of these quotients may always be set aside, provided that the computation be made to begin from 0 and 1. Again, if the number of quotients as they first come out be odd, the last quotient will lead us to a coefficient of a that stands in an odd place, whereas we want the last coefficient in an even place; therefore if the number of quotients be odd, both the first and last must be rejected; but if the number be even, then only the first quotient must be rejected, and the last must be reduced, that is, diminished by unity, as is always done in computiug the last equation, whether it happens in an even or an odd place, (see art. 175,) and the equation we are now enquiring af-

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ter will, in this case, be the last in the series. This known, if a series of numbers be computed from 0 and 1 by the help of the quotients still retained, the last term of the series will be r, and so the number $\frac{ra}{l} \times \overline{d-e} + d$ (which answers the conditions of the question) will be known.

If it be asked, what must be done when (in finding the greatest common measure l) the number of quotients is even, and the last is unity, since unity cannot be reduced; my answer is, that such a case can never happen.

EXAMPLE 1.

Let $a = 105$, $b = 40$, and consequently $l = 5$, as in the foregoing problem; and the quotients of a continual division from a and b will be 2, 1, 1, 1, 2; reject the first and last, because of their odd number, and with the remaining quotients 1, 1, 1 form from 0 and 1 the following series 0, 1, 1, 2, 3, and r will be the last term 3; whence $\frac{ra}{l} \times \overline{d-e} + d$, will be $\frac{3a}{5} \times \overline{d-e} + d = 63 \times \overline{d-e} + d$.

N. B. So long as the divisors a and b are in the same proportion one to the other, the expression of the number sought, to wit, $\frac{ra}{l} \times \overline{d-e} + d$ will be the same, let the divisors themselves be what they will; for the numbers $\frac{a}{l}$ and $\frac{b}{l}$ being the least in their proportion, must always be the same, and so will the quotients from which the number r is derived; all the difference then will only be the different restrictions to which the remainders d and e will be liable; for if this problem be possible, $\frac{d-e}{l}$ must always be a whole number, as was shewn before: thus if, instead of making the divisors 105 and 40, we had made them 21 and 8, the least in their proportion, the number $63 \times \overline{d-e} + d$ would still have satisfied the conditions of the problem; and then the remainders d and e might have been made any two whole numbers whatever, since $\frac{d-e}{1}$ will always be a whole number.

EXAMPLE 2.

Let a and b the given divisors be 840 and 36, whose greatest common measure l is 12, and the quotients in finding it 23 and 3, even in number;

number; drop therefore 23, and reduce the other quotient 3 to 2, and the ſeries from 0 and 1 will be 0, 1, 2; therefore in this caſe $r=2$, and $\frac{ra}{l}\times\overline{d-e}+d$ will be $140\times\overline{d-e}+d=141d-140e$, which is ſuch a number as the problem requires, as may be thus demonſtrated ſynthetically. Firſt I exterminate $-140e$, by obſerving that $140d-140e$, or $140\times\overline{d-e}$, or $1680\times\frac{d-e}{12}$ is a number diviſible by 840; therefore $140d-140e$ may be ſubtracted from $141d-140e$ without affecting the remainder; ſubtract it then, and there remains d: next I exterminate d thus; $141d-141e$, or $141\times\overline{d-e}$, or $1692\times\frac{d-e}{12}$ is a number diviſible by 36, and therefore $141d-141e$ may be ſubtracted from $141d-140e$ without affecting the remainder; ſubtract it therefore and you will have e remaining.

EXAMPLE 3.

Make $a=9$, $b=7$, and conſequently $l=1$, and the quotients in finding l will be 1, 3, 2, odd in number; drop the firſt and laſt, retaining only the middle quotient 3, and the ſeries will be 0, 1, 3; therefore $r=3$, and $\frac{ra}{l}\times\overline{d-e}+d=27\times\overline{d-e}+d=28d-27e$; and as the diviſors 9 and 7 are prime to each other, the ſynthetical demonſtration will be very eaſy; for if $28d$ be divided by 9, the remainder will be d, and if $-27e$ be divided by 9, the remainder will be 0; therefore $28d-27e$ divided by 9 will have d remaining: again, if $28d$ be divided by 7, the remainder will be nothing, and if $-27e$, that is, if $-28e+e$ be divided by 7, the remainder will be e; therefore $28d-27e$ divided by 7 will have e remaining: and this will always be the caſe when the diviſors d and e are prime to each other; that is, the truth of the canon may be ſhewn by a ſimple diviſion without any other artifice, the remainders in this caſe being ſubjected to no reſtriction.

EXAMPLE 4.

Let $a=30$, $b=10$, and conſequently $l=10$, and there will be but one quotient of a continual diviſion from 30 and 10, to wit 3; and this 3 muſt be rejected upon a double account, not only as it is the firſt quotient, but alſo as it is the laſt, ſince the number of quotients is an odd number; therefore to the two initial terms 0 and 1 of the ſeries no other terms can be added, becauſe there are no quotients whereby any other terms

can be generated; therefore in this case, the latter term 1 must be looked upon as the last term of the series. Here then $r=1$, and $\frac{ra}{l} \times \overline{d-e}+d$ $=3\times\overline{d-e}+d=4d-3e$, which I thus further demonstrate: $3d-3e$, or $30\times\frac{d-e}{10}$ is a multiple of 30; subtract then $3d-3e$ from $4d-3e$, and there remains d: again, $4d-4e$, or $40\times\frac{d-e}{10}$ is a multiple of 10; subtract then $4d-4e$ from $4d-3e$, and there remains e.

EXAMPLE 5.

Let $a=13$, and $b=7$, and consequently $l=1$; moreover let $e=0$; and the quotients of a continual division from 13 and 7 will be 1, 1, 6; drop the first and last on account of their odd number, and the remaining quotient will be 1, and the series will be 0, 1, 1; therefore $r=1$, and $\frac{ra}{l}\times\overline{d-e}+d=14d$, which will answer the conditions; for $14d$ divided by 13 will have d remaining, and $14d$ divided by 7 will have nothing remaining.

EXAMPLE 6.

Suppose the present year of Christ to be one thousand seven hundred thirty nine, and that there was a year not above two hundred years ago wherein the cycle of the sun was eight, and the cycle of the moon ten: What was the number of the year?

N. B. If to any year of Christ be added 9, and the sum be divided by 28, the remainder, or 28 if nothing remains, is what is called the cycle of the sun for that year; and if 1 be added to any year of Christ, and the sum be divided by 19, the remainder, or 19 if nothing remains, is the cycle of the moon: hence if any year of Christ be severally divided by 28 and 19, and the remainders be d and e respectively, $d+9$, or $d+9-28$ will be the cycle of the sun, and $e+1$, or $e+1-19$ will be the cycle of the moon for that year: now in our case, $d+9$ cannot be equal to 8, for then d would be negative; make then $d+9-28=8$, and you will have $d=27$; moreover make $e+1=10$, and you will have $e=9$; therefore the question is now reduced to this; *What number is that, which being severally divided by twentyeight and nineteen, will have twentyseven and nine respectively remaining?*

Here $a=28$, $b=19$, $l=1$, $d=27$, $e=9$; and the quotients of a continual division from 28 and 19 are 1, 2 and 9, in number odd; strike out

out therefore the first and last, and with the remaining quotient 2 compute the last term of the series 0, 1, 2, and you will have $r = 2$, and $\frac{ra}{l} \times \overline{d - e} + d = 56 \times \overline{d - e} + d$, or $57 \times \overline{d - e} + e$, which is a general rule for finding the year from the two cycles given: restore now the values of d and e, that is, make $d = 27$, and $e = 9$, and you will have $56 \times \overline{d - e} + d = 1035$, which number is too little for our purpose, and must be thus enlarged: the least common multiple of 28 and 19 is 532, in which revolution of years the cycles again return, and are both the same as they were 532 years before; add then one revolution of 532 years to 1035 the number above found, and the sum 1567 will be the true number of the year sought, as falling within the limits above prescribed: for $1567 + 9$, or 1576 being divided by 28, leaves 8 for the cycle of the sun; and $1567 + 1$, or 1568 being divided by 19, leaves 10 for the cycle of the moon.

PROBLEM 4.

207. *To find as many numbers as we please with this property, to wit, that if any one of them be severally divided by three given divisors* a, b *and* c, *whereof* a *is supposed the greatest and* c *the least, the remainders shall be three given numbers* d, e *and* f *respectively.*

SOLUTION.

Put A *for the least common multiple of* a *and* b; *then find (by the last problem) two numbers* g *and* h *of such a nature, that* g *being divided by* a *and* b *shall have* d *and* e *for remainders, and that* h *being divided by* A *and* c *shall have* g *and* f *for remainders; I say then that* h *will be one of the numbers that will answer the conditions: And when any one number of this kind is found, as many others as we please may be had by a continual addition and subtraction of the least common multiple of the three divisors* a, b *and* c, *found by art.* 172.

The demonstration of this solution is easy; for since by the supposition a measures A, and A measures $\overline{h - g}$, a measures $\overline{h - g}$; but a measures also $\overline{g - d}$; therefore a measures both $\overline{h - g}$ and $\overline{g - d}$, and consequently their sum $\overline{h - d}$: in like manner, since b measures A, and A measures $\overline{h - g}$, b measures $\overline{h - g}$; but b measures also $\overline{g - e}$; therefore b measures $\overline{h - e}$: therefore since a measures $\overline{h - d}$, and b measures $\overline{h - e}$, it follows that if h be severally divided by a and b, the remainders will be d and e respectively: but if h be divided by c, the remainder is

 f by

f by the ſuppoſition; therefore h is ſuch a number as being divided by the three given diviſors a, b and c, ſhall have the three given remainders d, e and f reſpectively. *Q. E. I.*

EXAMPLE I.

Let it be required to find a number, which being ſeverally divided by 105, 40 and 36, ſhall have d, e and f for remainders reſpectively.

Now before we enter upon the ſolution of this problem, or any other of this nature, we are in the firſt place to enquire what reſtriction they are liable to, in order to judge of the poſſibility or impoſſibility of ſuch problems: let ſuch an enquiry be made here, and we ſhall find that if this problem be poſſible, the remainders d, e and f muſt be ſo reſtrained, that $\frac{d-e}{5}$, $\frac{d-f}{3}$, and $\frac{e-f}{4}$ muſt all be whole numbers: this is evident from the ſolution of the foregoing problem; for 5 is the greateſt common meaſure of a and b, 3 that of a and c, and 4 that of b and c; let us therefore proceed upon theſe ſuppoſitions, and ſee whether by their help we cannot form and demonſtrate a general canon for the given diviſors without calling in any further aſſiſtance. Firſt then we are to find a number which being ſeverally divided by the two firſt diviſors 105 and 40, will have d and e for remainders; and ſuch a one we ſhall find the number $\overline{64d-63e}$, as in the firſt example to the foregoing problem; call this g: then as the leaſt common multiple of the two firſt diviſors 105 and 40 is 840, we are again to find another number which being divided by 840 and the third diviſor 36, will have g and f remaining; and ſuch a one we ſhall find the number $141g-140f$ to be, as in the ſecond example to the ſame problem: make therefore $64d-63e=g$, and $141g-140f=h$, and h will be ſuch a number as the problem requires.

If any one would have g ſtruck out of this canon, and ſo have the value of h expreſſed by the help of the remainders d, e and f only, this is eaſily done: for ſince $g=64d-63e$, we ſhall have $141g-140f=9024d-8883e-140f=h$; divide this number $9024d-8883e-140f$ by 2520, (which being the leaſt common multiple of all the three diviſors, cannot affect the remainders in reſpect of theſe diviſors,) and you will now have remaining a leſſer value of h, equal to $1464d-1323e-140f$; to this value add $2520e+2520f$ to make the parts all affirmative, and you will have $h=1464d+1197e+2380f$, which is the leaſt value of h that can be expreſſed in theſe general terms, till the remainders d, e and f come to be given in particular numbers.

The ſynthetical demonſtrations of theſe canons have ſomething very curious in them; and therefore I choſe this example as full as I could, that

that the general method of them might hereby be the better comprehended. We are then to demonſtrate that the number b above found, to wit, $1464d+1197e+2380f$ is ſuch a one as being ſeverally divided by 105, 40 and 36 will have d, e and f remaining. Now to ſhew this, I firſt divide b by 105, and find the remainder to be $99d+42e+70f$; but the further to exhauſt this remainder, in order to try whether the laſt remainder of this diviſion will be d, I expunge the other two quantities $42e$ and $70f$ by the help of thoſe two reſtrictions wherein the laſt remainder d is concerned, that is, by ſuppoſing both $\frac{d-e}{5}$ and $\frac{d-f}{3}$ to be whole numbers, expunging e by the former ſuppoſition wherein e is concerned, and f by the latter wherein f is concerned, thus: $42d-42e$, or $42\times\overline{d-e}$, or $210\times\frac{d-e}{5}$ is diviſible by 105 without any remainder; therefore $42d-42e$ may be added to $99d+42e+70f$ without affecting the ſubſequent remainders; add it then, and the ſum will be $141d+70f$: again, $70d-70f$, or $210\times\frac{d-f}{3}$ is diviſible by 105, and therefore $70d-70f$ may be added to $141d+70f$, and the ſum will be $211d$, which being divided by 105, there will remain d; therefore b is ſuch a number as being divided by 105 will have d for a remainder. In the next place I divide the number b, or $1464d+1197e+2380f$ by the next diviſor 40, and find the remainder to be $24d+37e+20f$, where the laſt remainder is to be e; therefore I exterminate the two quantities $24d$ and $20f$ by thoſe two reſtrictions wherein the laſt remainder e is concerned, that is, wherein it is ſuppoſed that both $\frac{d-e}{5}$ and $\frac{e-f}{4}$ are whole numbers, exterminating d by the former ſuppoſition wherein d is concerned, and f by the latter wherein f is concerned, thus: $24d-24e$, or $120\times\frac{d-e}{5}$ is diviſible by 40; therefore $24d-24e$ may be ſubtracted from $24d+37e+20f$, and the remainder will be $61e+20f$: again, $20e-20f$, or $80\times\frac{e-f}{4}$ is diviſible by 40; therefore $20e-20f$ may be added to $61e+20f$, and the ſum will be $81e$, which being divided by 40 will have e remaining; therefore b is ſuch a number as being divided by 40 will have e for a remainder. Laſtly I divide the number $1464d+1197e+2380f$ by 36, and the remainder is $24d+9e+4f$; but the laſt remainder is to be f; therefore I exterminate $24d$ and $9e$ by the help of the two reſtrictions wherein f is concerned, that is, by ſuppoſing both $\frac{d-f}{3}$ and

and $\frac{e-f}{4}$ to be whole numbers, exterminating d by the former ſuppoſition wherein d is concerned, and e by the latter wherein e is concerned, thus: $24d-24f$, or $72\times\frac{d-f}{3}$ is diviſible by 36; therefore $24d-24f$ may be ſubtracted from $24d+9e+4f$, and there will remain $9e+28f$: again, $9e-9f$, or $36\times\frac{e-f}{4}$ is diviſible by 36; therefore $9e-9f$ may be ſubtracted from $9e+28f$, and there remains $37f$, which being divided by 36, there will remain f: therefore $1464d+1197e+2380f$ is ſuch a number as being divided by 105, 40 and 36, will have for remainders d, e and f reſpectively. *Q. E. D.*

N. B. If the diviſors had been made 60, 45 and 36, the example would have been as full, and in leſs numbers.

EXAMPLE 2.

Let the diviſors be 9, 8 and 6: then becauſe the diviſors 9 and 8 are prime to each other, the reſtrictions will be reduced to theſe two, *viz.* that $\frac{d-f}{3}$ and $\frac{e-f}{2}$ muſt be both whole numbers; and the canon will be as follows: make $64d-63e=g$, and $13g-12f=h$, and h will be ſuch a number as is required: ſtrike g out of this canon, and you will have $832d-819e-12f=h$; divide by 72 (the leaſt common multiple of the three given diviſors) to reduce the expreſſion, and then add $72e+72f$ to take off all negation, and you will have $40d+45e+60f$ for the leaſt value of h that can be expreſſed in this form.

Now to try this theorem ſynthetically, I divide the number $40d+45e+60f$ by the firſt diviſor 9, and the remainder is $4d+6f$, where d is to be the laſt remainder; and therefore $6f$ muſt be baniſhed by the reſtriction which ſuppoſes $\frac{d-f}{3}$ to be a whole number, thus: $6d-6f$, or $18\times\frac{d-f}{3}$ is diviſible by 9; therefore $6d-6f$ may be added to $4d+6f$, and the ſum will be $10d$, which being divided by 9 leaves d. Then I proceed to the next diviſor 8, dividing $40d+45e+60f$ by 8, and the remainder is $5e+4f$, where e muſt be the laſt remainder; and therefore $4f$ muſt be baniſhed upon the ſuppoſition that $\frac{e-f}{2}$ is a whole number, thus: $4e-4f$, or $8\times\frac{e-f}{2}$ is diviſible by 8; therefore $4e-4f$

may

may be added to $5e+4f$, and the sum will be $9e$, which being divided by 8 leaves e for a remainder. Lastly, I divide $40d+45e+60f$ by the last divisor 6, and the remainder is $4d+3e$, both which quantities must be exterminated (because f is to be the last remainder) thus: $4d-4f$, or $12\times\frac{d-f}{3}$ is divisible by 6; therefore $4d-4f$ may be subtracted from $4d+3e$, and there will remain $3e+4f$; but $3e-3f$, or $6\times\frac{e-f}{2}$ is also divisible by 6; therefore $3e-3f$ may be subtracted from $3e+4f$, and there will remain $7f$, which being divided by 6 will have f for a remainder. *Q. E. D.*

EXAMPLE 3.

Let the divisors be 6, 5 and 4; and then the remainders will be liable to but one restriction, which is, that $\frac{d-f}{2}$ must be a whole number; and you will have the following canon, $25d-24e=g$, and $16g-15f=h$; whence exterminating g, you will have $400d-384e-15f=h$; divide by 60 (the least common multiple of the three given divisors) and to the remainder add $60e+60f$ to make the parts of that denomination affirmative, and you will have $40d+36e+45f=h$. To try this, divide $40d+36e+45f$ by the first divisor 6, and the remainder will be $4d+3f$; but $3d-3f$, or $6\times\frac{d-f}{2}$ is divisible by 6; therefore $3d-3f$ may be added to $4d+3f$, and the sum will be $7d$, which being divided by 6 leaves d for a remainder. Proceed now to divide $40d+36e+45f$ by the other two divisors 5 and 4 severally, and the remainders will come out e and f respectively, as will be found by a bare simple division only, without any other artifice.

From all these instances, and from the nature of the operations themselves we may see, that we shall never have a demand for more restrictions than those that are deduced at the beginning of the problem from the nature of the divisors; and that when no such restriction can be deduced, there will be no occasion for any, but the remainders will all come out by a simple division only, as will further appear in the next example.

EXAMPLE 4.

Let it be required to assign in what year of Christ the cycle of the sun was eight, the cycle of the moon ten, and the cycle of indiction ten.

Here

Here it muſt be known, that if to any year of Chriſt be added 3, and the ſum be divided by 15, the remainder, or 15 if nothing remains, is called the cycle of indiction for that year: hence, and from the definitions of the other two cycles already given in the laſt example of the laſt article it appears, that if any year of Chriſt be divided by 28, 19 and 15 ſeverally, and the remainders be d, e and f reſpectively, $d+9$, $e+1$ and $f+3$ will be the reſpective cycles of the ſun, moon and indiction; but if theſe numbers happen to be greater than 28, 19 and 15, then $d+9-28$, $e+1-19$, and $f+3-15$ will be the reſpective cycles: therefore we have three equations for determining d, e and f, to wit, $d+9-28=8$, $e+1=10$, and $f+3=10$; whence $d=27$, $e=9$, and $f=7$; ſo that upon the whole matter, we are *To find a number, which being divided by* 28, 19 *and* 15, *will have* 27, 9 *and* 7 *for remainders*; but we ſhall ſtill call the remainders d, e and f till we have computed a general canon for the numbers 28, 19 and 15, which canon can be under no reſtriction, nor ſubject to any exception, becauſe the numbers 28 and 19 are prime to each other, and both to the third number 15. Here then $g=57d-56e$, and $h=1065g-1064f=60705d-59640e-1064f$; divide this number by 7980 (the leaſt common multiple of the three given diviſors) and the remainder will be $4845d-3780e-1064f$; add $7980e+7980f$ to take off all negation, and you will have $4845d+4200e+6916f$ the leaſt of this form that will anſwer the conditions, and will equally ſerve to find any year of Chriſt for which the three cycles, and conſequently d, e and f are given, or it's place in the Julian period. As for example, inſtead of d, e and f ſubſtitute their values above found, to wit, 27, 9 and 7, and you will have $4845d+4200e+6916f=130815+37800+48412=217027$; which being divided by 7980 leaves 1567 for the year of Chriſt ſought; to which if 9, 1 and 3 be ſeverally added, and the ſums divided by 28, 19 and 15, the remainders will be 8, 10 and 10 the three cycles propoſed.

The Julian period is a revolution of 7980 *years, bearing date from before the beginning of the world, and is not yet paſſed over: It's character is, that if the number of any year of this period without any further preparation be divided by* 28, 19 *and* 15, *the remainders will be the three cycles of the ſun, moon and indiction for that year: therefore to find what place the year enquired after hath in this period, is nothing elſe but to find a number, which being divided by* 28, 19 *and* 15, *will have the three cycles* 8, 10 *and* 10 *for remainders.* Make then $d=8$, $e=10$ and $f=10$, and you will have $4845d+4200e+6916f=149920$; which being divided by 7980 leaves 6280; therefore the year ſought is the 6280th year of the Julian period, which anſwers to the 1567th year of Chriſt. Whence we may obſerve by the bye, that *If to any year of Chriſt be added* 4713,

the

the ſum will be the correſpondent year of the Julian period: the reaſon of this is, becauſe the difference betwixt 1567 and 6280 is 4713.

SCHOLIUM.

From what has been laid down in the two laſt problems, and eſpecially in this, I doubt not but the learner will be able to make his way through more complicated caſes, when more diviſors and more remainders are given, and conſequently more reſtrictions neceſſary; nor do I doubt but that he will be able to apply theſe reſtrictions ſo effectually, as to adapt ſynthetical demonſtrations to all caſes.

Theſe two laſt problems are ſuch as men of numbers have a curioſity to look into; and therefore I hope to be excuſed if I have dwelt ſomewhat longer than ordinary upon them; having however omitted ſeveral obſervations relating thereto, which to ſome perhaps would not have been unacceptable, purely becauſe I would not be too tedious to the generality of my readers.

PROBLEM 5.

208. *Of two given numbers* a *and* b, *whereof* a *is the greater, to find two multiples whoſe difference ſhall be any given number whatever that is diviſible by the greateſt common meaſure of* a *and* b.

SOLUTION.

Let d repreſent in general the difference of the multiples ſought; let $a=13$, and $b=7$, and their leaſt common multiple will be 91 by art. 171. This ſuppoſed, find by the third problem a number which being divided by a will have d remaining, and being divided by b will have nothing remaining; ſuch a one will be the number $14d$, as appears from the fifth example of that problem; therefore $13d$ and $14d$ are two multiples of a and b reſpectively, which will anſwer the conditions of the problem. Again, as 91, and conſequently $91d$ is a common multiple of both a and b, if $13d$, which is a multiple of a, be ſubtracted from $91d$, another multiple of a, the remainder $78d$ muſt alſo be a multiple of a by art. 169; and if $14d$, which is a multiple of b, be ſubtracted from $91d$, another multiple of b, the remainder $77d$ muſt alſo be a multiple of b; therefore $78d$ and $77d$ are other two multiples of a and b reſpectively, which will ſolve the problem as well as the former; the former ſolve it when a's multiple is intended to be leſs than that of b, the latter when a's multiple is intended to be greater than that of b, and both when the matter is left indifferent. But if we would reduce the multiples in both caſes to the leaſt in their kind, that is, if in both caſes we would find the leaſt multiples of a and b whoſe difference is d, the

 leſſer

leſſer multiples in both caſes, that is, $13d$ and $77d$ muſt be ſeverally divided by the leaſt common multiple of a and b, that is, by 91 : be it ſo, and let the remainders be m and n reſpectively : I ſay then, that m will be the leaſt multiple of a which with d added can make a multiple of b, and that n will be the leaſt multiple of b which with d added can make a multiple of a.

Firſt, that the remainder m is a multiple of a is plain, becauſe both the diviſor 91, and the dividend $13d$ were ſo. 2dly, That $m+d$ is a multiple of b is alſo evident ; for if 91 being taken from $13d$ as often as poſſible leaves m, the ſame 91 taken the ſame number of times from $14d$ will leave $\overline{m+d}$; and this $\overline{m+d}$ muſt be a multiple of b, becauſe both 91 and $14d$ were ſo. Laſtly, that the multiples m and $\overline{m+d}$ are the leaſt in their kind, appears from the nature of diviſion, and from art. 173 ; for the remainder m, if taken the leaſt poſſible, can never be greater than the diviſor, which was the leaſt common multiple of a and b, nor even equal to it, unleſs to prevent there being no remainder at all ; therefore the multiples m and $\overline{m+d}$ are the leaſt in their kind in the former caſe : and for the ſame reaſon the multiples n and $\overline{n+d}$ will be the leaſt in their kind in the latter. Thus we have found not only two ſorts of multiples which will equally ſolve the problem, but alſo the leaſt multiples of each ſort. *Q. E. I.*

N. B. 1ſt. If m, or n, or both be equal to nothing, their places muſt be ſupplied by the leaſt common multiple of a and b, as was hinted before. This can never happen but when either b, or a, or both meaſure d: if b meaſures d, and conſequently $13d$, then $13d$ will be a multiple of b ; but $13d$ is alſo a multiple of a by the conſtruction ; therefore $13d$ will be a common multiple of both a and b, and as ſuch, being divided by 91 their leaſt common multiple, will have no remainder ; therefore in this caſe the remainder m, if unſupplied, will be equal to nothing: in like manner, if a meaſures d, and conſequently $77d$, then $77d$ will be a common multiple of a and b, and ſo n will be equal to nothing ; therefore if both b and a meaſure d, both m and n will be equal to nothing.

2dly. If inſtead of the numbers 13 and 7 in the reſolution of this problem we had made uſe of any other numbers in the ſame proportion, as 39 and 21 whoſe greateſt common meaſure is 3, the canon adapted to theſe numbers would have been the ſame with the former, that is, the multiples of 39 and 21 would have been expreſſed by $13d$ and $14d$ in the former caſe, and by $78d$ and $77d$ in the latter ; but the ſynthetical demonſtration would have been ſomething different ; for in this caſe

d

$\frac{d}{3}$ muſt have been a whole number, and we muſt have argued thus: $13d$, or $\frac{39d}{3}$ is a multiple of 39, and $14d$, or $\frac{42d}{3}$ is a multiple of 21, and therefore theſe multiples will anſwer the condition: again, $78d$ and $77d$, or $\frac{234d}{3}$ and $\frac{231d}{3}$ are multiples of 39 and 21 reſpectively, becauſe the numbers 234 and 231 are ſo,

Of the ſolution here given take the following example. Let it be required to find two multiples of 13 and 7 whoſe difference ſhall be 500. Here $13d$ or 6500 being divided by 91 has 39 for a remainder; therefore $m=39$: again, $77d$ or 38500 divided by 91 has 7 for a remainder; therefore $n=7$; therefore 39 is the leaſt multiple of 13 which with 500 will make 539 a multiple of 7; and on the other hand, 7 is the leaſt multiple of 7 which with 500 will make 507 a multiple of 13.

PROBLEM 6.

209. *Having given two numbers* a *and* b *whoſe leaſt common multiple is* c, *and alſo a third number as* d *that is diviſible by the greateſt common meaſure of* a *and* b; *It is required to find, if poſſible, two multiples of* a *and* b *whoſe ſum ſhall be that given number* d, *or (which is the ſame thing) it is required to divide the given number* d *into two ſuch parts, that one part ſhall be a multiple of* a, *and the other a multiple of* b.

SOLUTION.

By the help of the laſt problem find m the leaſt multiple of a which with d added can make a multiple of b; and ſubtracting it from c the leaſt common multiple of a and b, call the remainder r; or if m before it's place be ſupplied, be equal to nothing, make $r=c$; then if r be greater than or equal to d, the problem will be incapable of any ſolution; but if r be leſs than d, I ſay then that r will be the leaſt multiple of a, and $\overline{d-r}$ the greateſt multiple of b that will ſolve the problem: and this firſt ſolution being obtained, if the problem admits of more, they will eaſily be had by a continual addition and ſubtraction of the leaſt common multiple c, as $\overline{r+c}$ and $\overline{d-r-c}$, $\overline{r+2c}$ and $\overline{d-r-2c}$, $\overline{r+3c}$ and $\overline{d-r-3c}$, &c: therefore if $\overline{d-r}$ in the firſt ſolution be divided by c ſo as to have any remainder as s, the quotient which ſhews how often c can be ſubtracted from $\overline{d-r}$, will alſo ſhew how many ſolutions the problem will admit of after the firſt; and therefore this quotient in-

 creaſed

creaſed by unity will be the number of ſolutions in the whole; laſtly I ſay that the number s will be the leaſt multiple of b, and $d-s$ the greateſt multiple of a that will ſolve the problem this way: ſo that the numbers r and $\overline{d-r}$, as alſo $\overline{d-s}$ and s may be called the extreme ſolutions; except when the problem admits of but one ſolution, and then theſe extremes will unite in that ſingle ſolution.

DEMONSTRATION.

1ſt. Since by the conſtruction both c and m are multiples of a, their difference $\overline{c-m}$ or r will be a multiple of a by art. 169; and ſince both c and $\overline{m+d}$ are multiples of b, their difference $\overline{m-c+d}$ or $\overline{d-r}$ will be a multiple of b.

2dly. When $\overline{d-r}$ was divided by c, the remainder was s; therefore c meaſures $\overline{d-r-s}$: ſince then b meaſures c, and c meaſures $\overline{d-r-s}$, b meaſures $\overline{d-r-s}$; but b meaſures $\overline{d-r}$ as was proved before; therefore b meaſures both $\overline{d-r}$ and $\overline{d-r-s}$, and conſequently their difference s; therefore s is a multiple of b: again, ſince a meaſures c, and c meaſures $\overline{d-r-s}$, a meaſures $\overline{d-r-s}$; but a meaſures r, as was proved before; therefore a meaſures both r and $\overline{d-r-s}$, and conſequently their ſum $\overline{d-s}$; therefore $\overline{d-s}$ is a multiple of a.

3dly. We are in the laſt place to prove that r and $\overline{d-r}$, as alſo $\overline{d-s}$ and s are extreme ſolutions: and firſt, if any ſolution of this problem could be obtained wherein any leſs multiple of a than r was concerned, it muſt be had by ſubtracting the leaſt common multiple c from r, and adding it to $d-r$, and then the multiples would be $\overline{r-c}$ and $\overline{d-r+c}$; but r or $\overline{c-m}$ can never be greater than c, and therefore $r-c$ muſt either be nothing or negative, both which caſes are excluded out of the problem; therefore r muſt be the leaſt multiple of a, and $d-r$ the greateſt multiple of b that will ſolve the problem: and by a like proceſs it may be demonſtrated that s is the leaſt multiple of b, and $\overline{d-s}$ the greateſt multiple of a that will ſolve the problem.

An example to the foregoing ſolution.

Let it be required to divide the number 500 into two ſuch parts that one part may be a multiple of 13, and the other of 7. Here $a=13$, $b=7$, $c=91$, and $d=500$, as in the laſt problem; and ſince m was

there-

there found equal to 39, we ſhall have $\overline{c-m}$ or $r=52$, and $\overline{d-r}=448$; therefore 52 and 448 are the firſt numbers that will ſolve the problem, the former being a multiple of 13, and the latter of 7: divide 448 by 91, and the quotient will be 4, and the remainder 84; therefore 416 which is a multiple of 13, and 84 which is a multiple of 7 are the laſt numbers that will ſolve the problem. The ſolutions are five in number, and being placed in order will ſtand thus, the firſt number of every ſet being a multiple of 13, and the ſecond of 7.

1ſt.	52	and	448,
2d	143	and	357,
3d.	234	and	266,
4th.	325	and	175,
5th.	416	and	84.

LEMMA 12.

A THEOREM.

210. *Let* $\frac{a}{b}$ *expreſs any fraction in it's leaſt terms, and let this fraction be multiplied by ſome whole number* d, *ſo that the product* $\frac{ad}{b}$ *may be alſo a whole number: I ſay then that the multiplicator* d *muſt either be equal to, or ſome multiple of the denominator* b.

For ſince by the ſuppoſition, the fraction $\frac{ad}{b}$ is equivalent to ſome whole number, let that number be c; then we ſhall have $c=\frac{ad}{b}$, and $\frac{c}{d}=\frac{a}{b}$; therefore the fraction $\frac{c}{d}$ muſt either be the ſame with the fraction $\frac{a}{b}$, or elſe it muſt be reducible to it, as to it's leaſt terms, by the greateſt common meaſure of c and d, ſince $\frac{a}{b}$ is ſuppoſed to be in it's leaſt terms: if $\frac{c}{d}$ be the ſame with $\frac{a}{b}$, then d muſt be the ſame with b, which is one caſe of the lemma; if $\frac{c}{d}$ be equal to, but not the ſame with $\frac{a}{b}$, let e be the greateſt common meaſure of c and d; then will $\frac{c}{e}=a$; and $\frac{d}{e}=b$, and $d=eb$, in which caſe d is a multiple of b. *Q. E. D.*

COROL-

COROLLARY.

Hence I infer, that if $\frac{a}{b}$ *expresses any fraction in it's least terms, the denominator* b *will be the least number that fraction can be multiplied by to reduce it to a whole number.*

PROBLEM 7.

211. *Let* a, b, c, d, e, &c *be a number of hands like those of a clock, all turning uniformly upon the same center, and all moving the same way; and let the same letters* a, b, c, d, e, &c *represent also their respective periodical times, or numbers proportionable to them: What will be the synodical period of the whole system, that is, supposing these hands to start all together from the same point as* s, *how long will it be before they all come together again for the first time, whether this conjunction happens in the point* s, *or in any other part of the circle wherein these motions are performed?*

N. B. To say that of two hands *a* and *b*, *a* is got one circle and a half before *b*, or two circles and a half, or three circles and a half, is to say no more than that *a* is half a circle before *b*; and to say that *a* is one circle or two circles or three circles before *b*, is the same thing as to say that the two hands *a* and *b* are coincident: *And whenever any angular distance of this kind is expressed by a mixt number, or by an improper fraction, to reduce it* (a phrase frequently made use of in the following solution) *implies two things; first to throw away the integral part, and then to reduce the remaining fraction into it's least terms:* thus the distance $3\frac{4}{6}$ after reduction becomes $\frac{2}{3}$; for it is the same thing to say that *a* is three circles and four sixth parts of a circle before *b*, as to say that *a* is two thirds of a circle before *b*, or one third behind it. This language being well understood and attended to, the solution will be as follows.

SOLUTION.

1st. Let *p* be the synodical period of the two first hands *a* and *b*: then to find the distance of the hand *a* from the point *s* at the end of the time *p* say, If in the time *a* the first hand makes one revolution, how many revolutions and parts of a revolution will it make in the time *p*? and the answer is $\frac{p}{a}$; therefore the distance of the hand *a* from the point *s* at the end of the time *p* is $\frac{p}{a}$; and for the same reason the distance of the

hand

hand b from the point s is $\frac{p}{b}$; therefore the difference of these two fractions, to wit, $\frac{p}{a}-\frac{p}{b}$ shews how much the hand a is advanced before the hand b during the time p; but after the two hands a and b have departed from the point s, and parted from one another, they can never come together again till one is advanced an entire circle before the other; therefore if p be the synodical period of the two hands, we shall have $\frac{p}{a}-\frac{p}{b}=1$, and $p=\frac{ab}{b-a}$: this is upon a supposition that the hand a is the swifter mover; but if b be the swifter mover, we shall then have $\frac{p}{b}-\frac{p}{a}=1$, and $p=\frac{ab}{a-b}$; universally, p is the product of the two numbers a and b divided by their difference. And if p be the time at the end of which the two hands a and b will come first together, at the end of the time $2p$ they will come together a second time, and at the end of the time $3p$ a third time, and so on.

2dly. At the end of the time p, the distance of the hand a, and consequently of both the hands a and b from the point s will be $\frac{p}{a}$, as before; and at the same time the distance of the hand c from the same point s will be $\frac{p}{c}$; reduce therefore these two distances $\frac{p}{a}$ and $\frac{p}{c}$, and then taking their difference, reduce that also to it's least terms $\frac{n}{q}$; and then the distance of the hand c from the two coincident hands a and b at the end of the time p will be $\frac{n}{q}$, and at the end of the time $2p$, $\frac{2n}{q}$, and at the end of the time $3p$, $\frac{3n}{q}$, &c, and therefore at the end of the time pq the distance of the hand c from the coincident hands a and b will be n; but n is a whole number, and implies a whole number of circles or revolutions; therefore at the end of the time pq the hand c will coincide with the other two a and b: and that this will be the first coincidence of the three hands a, b and c since their departure from s, is evident from the last article, the denominator q being the least number the fraction $\frac{n}{q}$ can be multiplied by to make it a whole number; therefore the time pq is the synodical period of the three first hands a, b and c.

3dly,

3dly. At the end of the time pq, the diſtance of the hand a, and conſequently of all the three hands a, b and c from the fixed point s is $\frac{pq}{a}$, and the diſtance of the hand d from the ſame is $\frac{pq}{d}$; reducing then the diſtances $\frac{pq}{a}$ and $\frac{pq}{d}$, let their difference after reduction be a fraction whoſe denominator is r when reduced to it's leaſt terms, and the time pqr will be the ſynodical period of the four firſt hands a, b, c and d.

4thly. At the end of the time pqr the diſtance of a from s will be $\frac{pqr}{a}$, and the diſtance of e from s will be $\frac{pqr}{e}$, and if theſe diſtances be reduced, and their difference be alſo reduced to it's leaſt denominator s, the time $pqrs$ will be the ſynodical period of the five firſt hands a, b, c, d, e: and ſo on. The ſolution then (to recapitulate) is as follows:

1ſt. *Multiply the two periodical times* a *and* b *together, and then dividing their product by their difference, call the quotient* p, *whether it be a whole number or a fraction.*

2dly. *Reduce the two diſtances* $\frac{\mathrm{p}}{\mathrm{a}}$ *and* $\frac{\mathrm{p}}{\mathrm{c}}$, *and then taking their difference, reduce that alſo to it's leaſt denominator* q.

3dly. *Reduce the diſtances* $\frac{\mathrm{pq}}{\mathrm{a}}$ *and* $\frac{\mathrm{pq}}{\mathrm{d}}$, *and their difference to it's leaſt denominator* r.

4thly. *Reduce the diſtances* $\frac{\mathrm{pqr}}{\mathrm{a}}$ *and* $\frac{\mathrm{pqr}}{\mathrm{e}}$, *and their difference to it's leaſt denominator* s; &c: *I ſay then that* p *will be the ſynodical period of the two firſt hands,* pq *that of the three firſt,* pqr *that of the four firſt,* pqrs *that of the five firſt,* &c.

N. B. It matters nothing, as to the concluſion, what order the hands are taken in.

EXAMPLE.

Let $a=3$, $b=7$, $c=10$, $d=12$, $e=15$.

SOLUTION.

1ſt. $p=\frac{21}{4}$; therefore $\frac{p}{a}$ is $\frac{7}{4}$, and when reduced $\frac{3}{4}$; $\frac{p}{c}$ is $\frac{21}{40}$, and the difference between $\frac{3}{4}$ and $\frac{21}{40}$ is $\frac{9}{40}$; therefore $q=40$.

2dly.

2dly. Therefore $pq = \frac{21}{4} \times 40 = 210$; therefore $\frac{pq}{a}$ is 70, and when reduced 0; $\frac{pq}{d}$ is $\frac{210}{12}$, and when reduced $\frac{1}{2}$, and the difference betwixt 0 and $\frac{1}{2}$ is $\frac{1}{2}$; therefore r is 2.

3dly. Therefore $pqr = 420$; therefore $\frac{pqr}{a}$ is 140, and when reduced 0; $\frac{pqr}{e}$ is 28, and when reduced 0, and the difference between 0 and 0 is 0, or $\frac{0}{1}$; therefore s is 1. So that the synodical period of the two first hands is $\frac{21}{4}$, that of the three first 210, that of the four first 420, and that of the five first 420; which shews that when d joined the three first hands a, b and c, e joined them at the same time.

If the hand b should move a contrary way to that of a, then the equation which determines p will not be $\frac{p}{a} - \frac{p}{b} = 1$ as before, but $\frac{p}{a} + \frac{p}{b} = 1$; whence p will be found equal to $\frac{ab}{a+b}$; for the hands a b must now be looked upon as lying on contrary sides of the point s, in which case the sum of their distances from this point will be their distance from each other: again, if c moves contrary to a, the distances $\frac{p}{a}$ and $\frac{p}{c}$ must be reduced as before; but then it must be their sum, and not their difference, which being reduced to it's least terms will give the denominator q; and so on.

PROBLEM 8.

212. *Let there be three unknown quantities* x, y *and* z, *whose relation to each other is expressed by the two following equations,*

$$x + 2y + 3z = 20, \quad \textit{and}$$
$$4x + 5y + 6z = 47:$$

It is required to find the values of x, y *and* z *in integral and affirmative numbers.*

SOLUTION.

Equation 1st, $x + 2y + 3z = 20$.

2d, $4x + 5y + 6z = 47$.

Subtract the second equation from four times the first, and you will have $3y + 6z = 33$; whence

Equation 3d, $y = 11 - 2z$.

 Subtract

Subtract five times the first equation from twice the second, and you will have $3x-3z=-6$, and

Equation 4th, $x=z-2$.

And thus we have the values of x and y with respect to z, and z itself is left undetermined because there were but two equations to three unknown quantities: and that these are the true values of x and y will sufficiently appear by substituting $z-2$ and $11-2z$ instead of x and y in the first and second equations; for upon that supposition we shall have in the first equation $x+2y+3z=z-2+22-4z+3z=20$, and in the second equation we shall have $4x+5y+6z=4z-8+55-10z+6z=47$. Now as to the values of x, y and z, it appears from the third and fourth equations that whatever whole number is substituted for z, x and y must necessarily come out whole numbers; for if z be a whole number, not only $z-2$ or x will be a whole number, but also $11-2z$ or y; therefore if the problem was subject to no other limitations than barely that x, y and z should be whole numbers, the number of solutions would be infinite: but besides their being whole numbers, it is particularly specified in the problem that they shall be affirmative numbers; and therefore to find out the number of solutions in this case, we must again attend to the values of x and y in the two last equations: now since in the fourth equation $x=z-2$, it is plain that if x be affirmative, z must be taken greater than 2: again, since in the third equation $y=11-2z$, if y be affirmative, $2z$ must be less than 11, and z less than $5\frac{1}{2}$; whence it follows *a fortiori* that z must be less than 6; therefore that x and y as well as z may be affirmative, the value of z must consist between these two limits, *viz.* 2 and 6; therefore if all the three quantities x, y and z be affirmative, there are but three numbers that can be substituted for z, to wit, the numbers 3, 4 and 5; so that by the last limitation of the problem, all the infinite number of solutions it would otherwise have admitted of, are now reduced to three, and are thus determined: Since any of the three numbers 3, 4 and 5 may be put for z, let $z=3$; then we shall have $z-2$ or $x=1$, and $11-2z$ or $y=5$; so that the numbers x, y and z will be 1, 5 and 3 respectively, and will answer the conditions of the first and second equations; for upon this supposition $x+2y+3z=1+10+9=20$, and $4x+5y+6z=4+25+18=47$. Let $z=4$; then we have $z-2$ or $x=2$, $11-2z$ or $y=3$, and the numbers will be 2, 3 and 4, which will also answer; for in this case $x+2y+3z=2+6+12=20$, and $4x+5y+6z=8+15+24=47$. Let $z=5$; then will $z-2$ or $x=3$, $11-2z$ or $y=1$, and the numbers will be 3, 1 and 5; here then again we have $x+2y+3z=3+2+15=20$, and $4x+5y+6z=12+5+30=47$.

x	y	z.
1	5	3.
2	3	4.
3	1	5.

LEMMA 13.

A PROBLEM.

213. *Let* a, b *and* c *be three fixed or determinate whole numbers, whereof* a *and* b *are prime to each other; and let* x *and* y *be any two variable or indeterminate whole numbers, whose relation to each other is constantly expressed by the equation* $ax \pm by = c$: *What will be the next whole number values of* x *and* y, *whose relation to each other can be expressed by the same equation?*

CASE 1.

Let the equation be $ax - by = c$, and let the new values of x and y be $x+d$ and $y+e$; then since the relation of these two values is to be expressed by the same equation as was that of the former, the equation will be $ax+ad-by-be=c$; but $ax-by=c$ *ex hypothesi*; therefore $ad-be=0$, and $be=ad$, and $\frac{e}{d}=\frac{a}{b}$; therefore if e and d be any two numbers taken in the same proportion as a to b, the values $x+d$ and $y+e$ will fall under the same relation with x and y: but to find the nearest values to x and y, e and d must not only be in the same proportion with a and b, but they must be the least in their proportion, that is, the fraction $\frac{e}{d}$ must be in it's least terms: since then the two fractions $\frac{a}{b}$ and $\frac{e}{d}$ are equal in value, and both in their least terms, they must also be equal in terms, that is, e must be equal to a, and d to b; therefore the nearest values to x and y in the same relation are $x+b$ and $y+a$, or $x-b$ and $y-a$.

CASE 2.

Let the equation be $ax+by=c$, and let $x+d$ and $y+e$ be the nearest values of x and y in the same relation, and the equation will be $ax+ad+by+be=c$; but $ax+by=c$ *ex hypothesi*; therefore $ad+be=0$; therefore $be=-ad$, and $\frac{e}{d}=-\frac{a}{b}$; therefore if e and d

 be

be the leaſt in their proportion, as they ought to be, either e muſt be equal to $-a$ and d to b, or e to $+a$ and d to $-b$; therefore the two neareſt values to x and y in the ſame relation will be $x+b$ and $y-a$, or $x-b$ and $y+a$.

COROLLARY.

Hence, if the values of x *be taken in an arithmetical progreſſion whoſe common difference is* b *the coefficient of* y, *the reſpective values of* y *will alſo form an arithmetical progreſſion whoſe common difference is* a *the coefficient of* x: *and if the equation be* $ax-by$ &c, *both progreſſions will increaſe, or both decreaſe together; but if the equation be* $ax+by$ &c, *and if the values of* x *form an increaſing progreſſion, thoſe of* y *will form a decreaſing one, and* vice verſa.

PROBLEM 9.

214. *It is required to divide a hundred into three ſuch parts* x, y *and* z, *that* $9x+15y+20z$ *may make fifteen hundred.*

SOLUTION.

Equation 1ſt, $x+y+z=100$.
2d, $9x+15y+20z=1500$.

Subtract nine times the firſt equation from the ſecond, and you will have $6y+11z=600$; whence

Equation 3d, $y=100-\frac{11z}{6}$.

Subtract the ſecond equation from fifteen times the firſt, and you will have $6x-5z=0$; whence

Equation 4th, $x=\frac{5z}{6}$.

And if theſe values of x and y in the two laſt equations be ſubſtituted inſtead of x and y in the two firſt, they will anſwer the conditions of thoſe equations, as will eaſily appear upon tryal.

Now as to the determination of the values of x and y, where $x=\frac{5z}{6}$ and $y=100-\frac{11z}{6}$, it is evident that if for z be ſubſtituted any number that is diviſible by 6 without a remainder, that is, any term of this arithmetical progreſſion 6, 12, 18, 24, *&c ad infinitum*, the quantity x will come out a whole number and affirmative; but as $y=100-\frac{11z}{6}$, it is evident that y cannot be affirmative unleſs $\frac{11z}{6}$ be leſs than 100,

100, that is, unlefs $11z$ be lefs than 600, or z lefs than $54\frac{6}{11}$; therefore z muft not be made equal to any whole number that is greater than 54; therefore z may be made equal to any multiple of 6 from 6 to 54 inclufively. Now as the equation expreffing the relation betwixt x and z was $6x - 5z = 0$, it follows from the laft article, that if the values of z be taken in an increafing arithmetical progreffion whofe common difference is 6, the correfpondent values of x will form an increafing arithmetical progreffion whofe common difference is 5: again, fince the equation expreffing the relation betwixt y and z was $6y + 11z = 600$, it follows that if the values of z be made to increafe in an arithmetical progreffion whofe common difference is 6, thofe of y will fink in an arithmetical progreffion whofe common difference is 11. Make $z = 6$, and $100 - \frac{11z}{6}$ or y will be 89, and $\frac{5z}{6}$ or x will be 5, and the three parts fought will be 5, 89 and 6, which will anfwer the conditions.

As to the other folutions, (for they are nine in all,) thefe are eafily found by the obfervations above, and are put down in the following table:

x	y	z.
5	89	6.
10	78	12.
15	67	18.
20	56	24.
25	45	30.
30	34	36.
35	23	42.
40	12	48.
45	1	54.

PROBLEM 10.

215. *Let it be required to divide the number twentyfour into three fuch parts* x, y *and* z, *that* $x + 8y + 12z$ *may make two hundred and one.*

SOLUTION.

Equation 1ft, $x + y + z = 24$.

2d, $x + 8y + 12z = 201$.

Subtract the firft equation from the fecond, and you will have $7y + 11z = 177$; whence

Equation 3d, $y = \frac{177 - 11z}{7}$.

Subtract

Subtract the ſecond equation from eight times the firſt, and you will have $7x - 4z = -9$; whence

Equation 4th, $x = \frac{4z - 9}{7}$.

From the third and fourth equations it appears, that to make x and y whole numbers as well as z, ſome ſuch number muſt be ſubſtituted for z as will make not only $\frac{177 - 11z}{7}$, but alſo $\frac{4z-9}{7}$ a whole number: now had both $\frac{177}{7}$ and $\frac{9}{7}$ been whole numbers, this problem might have been ſolved as was the laſt; but as it happens otherwiſe, we muſt proceed after another manner; but firſt we ought to enquire whether theſe two conditions, *viz.* that both $\frac{177 - 11z}{7}$ and $\frac{4z-9}{7}$ are whole numbers, are conſiſtent with each other or not; for if they be found inconſiſtent, then it will be impoſſible for both x and y together to be whole numbers, and the problem will be impoſſible: divide then $4z - 9$ by 7, which in the preſent caſe is no more than ſubtracting 7 from $4z - 9$, and the remainder will be $4z - 2$; divide alſo $177 - 11z$ by 7, and the remainder will be $2 - 4z$; and therefore the enquiry is now reduced to this, whether it be poſſible for $\frac{4z-2}{7}$ and $\frac{2-4z}{7}$ to be both whole numbers; and this ſuppoſition will be ſo far from being impoſſible, that one part neceſſarily implies the other, theſe two numbers being the ſame affirmatively and negatively taken, and therefore they muſt either be both or neither of them whole numbers. This matter being ſettled, we are in the next place to enquire, what will be the leaſt number which being put for z will make $\frac{4z-2}{7}$ a whole number: now in order to this, make 7 the denominator equal to a, and 4 the coefficient of z in the numerator equal to b, and from theſe values of a and b derive a ſeries of equations as in the 175th article, and they will be as follows:

$$1a - 0b = +7.$$
$$0a - 1b = -4.$$
$$a - b = +3.$$
$$a - 2b = -1.$$
$$3a - 5b = +1.$$

Of the two laſt equations take that whoſe abſolute term is negative, to wit, $a - 2b = -1$, becauſe -2 the numeral part of the numerator

$4z - 2$

$4z-2$ is negative; and then multiplying the equation $a-2b=-1$ by 2 that numeral part taken affirmatively, you will have $2a-4b=-2$ the ſimpleſt equation of it's kind: make 4 the coefficient of $-b$ in this laſt equation equal to z, and reſtoring now the values of a and b, to wit, 7 and 4, the equation will be changed into this, $14-4z=-2$, and you will have $\frac{4z-2}{7}=2$, a whole number; therefore if the whole number 4 be put for z, the other two parts x and y will alſo be whole numbers, and you will have x or $\frac{4z-9}{7}=1$, and y or $\frac{177-11z}{7}=19$, and therefore the parts x, y and z will be 1, 19 and 4 reſpectively. Now to know in what progreſſions the values of x, y and z will vary, ſince the equation between x and z was $7x-4z=-9$, it follows from art. 213, that if the values of z be continually increaſed by 7, the values of x will be continually increaſed by 4; and ſince the equation between y and z was $7y+11z=177$, it follows alſo that if the values of z be continually increaſed by 7, thoſe of y will be continually diminiſhed by 11; therefore ſince the firſt value of x is 1; it's ſeveral values will be 1, 5, 9, 13, &c; the ſeveral values of y will be 19, 8, -3, -14, &c; and laſtly thoſe of z will be 4, 11, 18, 25, &c: whence it appears that this problem is not capable of above two ſolutions, becauſe if the ſolutions be continued any further, y will be negative: here then we have $x=1$, $y=19$, $z=4$; or $x=5$, $y=8$, and $z=11$.

PROBLEM II.

216. *Suppoſe one would buy forty birds, conſiſting of partridges, larks and quails for ninetyeight pence, paying three pence apiece for the partridges, halfpence apiece for the larks, and four pence apiece for the quails: The queſtion is, how many he muſt have of each ſort.*

SOLUTION.

Put x for the number of partridges, y for the number of larks, and z for the number of quails; then will $3x$, $\frac{y}{2}$, and $4z$ expreſs the number of pence to be given for each ſort; and the problem abſtracted from words will ſtand thus: *If* $x+y+z=40$, *and* $3x+\frac{y}{2}+4z=98$, *or* $6x+y+8z=196$; *what are* x, y *and* z?

Equation 1ſt, $x+y+z=40$.

2d, $6x+y+8z=196$.

Subtract

Subtract the ſecond equation from ſix times the firſt, and you will have $5y - 2z = 44$; whence

Equation 3d, $y = \frac{2z + 44}{5}$.

Subtract the firſt equation from the ſecond, and you will have $5x + 7z = 156$; whence

Equation 4th, $x = \frac{156 - 7z}{5}$; therefore $\frac{2z + 44}{5}$ and $\frac{156 - 7z}{5}$ muſt both be whole numbers, the poſſibility whereof I thus demonſtrate: divide $156 - 7z$ by 5, and the remainder will be $1 - 2z$; divide $2z + 44$ by 5, and the remainder will be $2z + 4$, from which again ſubtracting 5, the remainder will be $2z - 1$; compare then theſe two remainders $2z - 1$ and $1 - 2z$, and the queſtion will now be reduced to this, *viz.* whether if $\frac{2z - 1}{5}$ be a whole number, $\frac{1 - 2z}{5}$ can be ſo too; and the anſwer is ready, that if the former be a whole number, the latter muſt be ſo, ſince it is but the negative of the former: therefore if $\frac{2z + 44}{5}$ be a whole number, $\frac{156 - 7z}{5}$ muſt be ſo too. This being determined, let us now reſume one of the former remainders, to wit, $2z - 1$, and let us enquire into the leaſt number which being ſubſtituted for z will make $\frac{2z - 1}{5}$ a whole number: and this might be found out after the ſame manner as in the laſt problem; but becauſe the denominator 5 is but a ſmall number, I rather chuſe to do it by tryal (ſince four tryals at moſt muſt determine the queſtion,) that is, I make z equal to 1, 2, 3 and 4 ſucceſſively, and try in which of theſe caſes $\frac{2z - 1}{5}$ will be a whole number, and I find it will ſucceed when z is made equal to 3; for then $\frac{2z - 1}{5} = 1$.

Note. The number of tryals muſt always be leſs by unity than the denominator.

Thus then we have found $z = 3$; whence y or $\frac{2z + 44}{5}$ will be 10, and x or $\frac{156 - 7z}{5}$ will be 27; therefore if he buys twentyſeven partridges, ten larks, and three quails, he will have forty birds for ninety-eight pence, as the problem requires.

As

As to the other ſolutions, ſince the equation for x and z was $5x+7z=156$, and the equation for y and z was $5y-2z=44$, it follows from art. 213, that the ſucceſſive values of z muſt be increaſed by a continual addition of 5, thoſe of y by a continual addition of 2, and thoſe of x muſt be diminiſhed by a continual ſubtraction of 7; whence we have the following ſolutions:

x	y	z.
27	10	3.
20	12	8.
13	14	13.
6	16	18.

Beſides the four ſolutions of this problem exhibited in the table it will be impoſſible to aſſign any others; for ſhould this table be carried on but one degree lower, the value of x would be found to be -1; and ſhould it be taken but one degree higher, the value of z would be found to be -2.

In my ſolution of this problem I left the value of the laſt quantity z undetermined, and reduced thoſe of x and y to that, as uſual: but if I had left the middle quantity y undetermined, and had reduced the values of x and z to the quantity y, the ſolution in this particular caſe would have been ſomewhat eaſier, and might have been obtained in the ſame manner as in the laſt problem but one, thus: ſubtract the ſecond equation from ſix times the firſt, as they ſtand in the foregoing ſolution, and you will have $5y-2z=44$; whence $z=\frac{5y}{2}-22$; therefore if z be affirmative, $\frac{5y}{2}$ muſt be greater than 22, and $5y$ greater than 44, and y greater than $8\frac{4}{5}$; therefore if z be affirmative, y muſt be greater than 8: again, ſubtract the ſecond equation from eight times the firſt, and you will have $2x+7y=124$, and $x=62-\frac{7y}{2}$; therefore $\frac{7y}{2}$ muſt be leſs than 62, and $7y$ leſs than 124, and y leſs than $17\frac{5}{7}$; therefore if x be affirmative, y muſt be leſs than 18; therefore all the values of y muſt lie between 8 and 18: but $\frac{7y}{2}$ in the expreſſion of x, and $\frac{5y}{2}$ in the expreſſion of y ſhew that $\frac{y}{2}$ muſt be a whole number, or (which comes to the ſame thing) that y muſt be an even number; therefore all the even numbers between 8 and 18 may be put for y, to wit, 10, 12, 14, 16: let $y=10$, and we ſhall have x or $62-\frac{7y}{2}=27$, and z or $\frac{5y}{2}-22=3$; and hence may all the other values of x and z be found as in the foregoing table.

PROBLEM 12.

217. *Suppose one would buy twenty birds for twenty pence, to wit, ducks at twopence apiece, partridges at halfpence apiece, and geese at threepence apiece: How many must he have of each sort?*

SOLUTION.

Put x, y and z for the number of ducks, partridges and geese respectively, and consequently $2x$, $\frac{y}{2}$ and $3z$ for the price of each sort in pence, and you will have these two fundamental equations; $x+y+z=20$, and $2x+\frac{y}{2}+3z=20$, or $4x+y+6z=40$; whence

Equation 1st, $x+y+z=20$.
2d, $4x+y+6z=40$.

Expunge z by subtracting the second equation from six times the first, and you will have $2x+5y=80$, and

Equation 3d, $x=40-\frac{5y}{2}$.

Therefore $\frac{5y}{2}$ must be less than 40, and $5y$ less than 80, and y less than 16. Again, expunge x by subtracting the second equation from four times the first, and you will have $3y-2z=40$, and

Equation 4th, $z=\frac{3y}{2}-20$; whence $\frac{3y}{2}$ must be greater than 20, and $3y$ greater than 40, and y must be greater than 13, and less than 16: but $\frac{5y}{2}$ in the third equation, and $\frac{3y}{2}$ in the fourth shew that y must be an even number; and there is but one even number between 13 and 16, to wit 14; therefore y must be 14, and the problem will admit but of one solution; therefore x or $40-\frac{5y}{2}$ must be 5, and z or $\frac{3y}{2}-20$ must be 1; that is, there must be 5 ducks, 14 partridges, and 1 goose.

Monsieur *Bachet* in his comment upon the forty first proposition of the fourth book of *Diophantus*, cites an epigram containing this problem, which (such as it is) I have here transcribed.

Ut tot emantur aves, bis denis utere nummis;
Perdix, anser, anas empta vocetur avis:
Sit simplex obolus pretium perdicis, ematur
Sex obolis anser, bisque duobus anas.

Ut

Ut tua procedat in lucem quæſtio, mentem
Conſule; ſic loquitur pectoris arca mihi:
Sint anates tres atque duæ, ſimplex erit anſer,
Accipe perdices quatuor atque decem.

PROBLEM 13.

218. *Twenty perſons conſiſting of men, women and children at a collation paid twenty ſhillings, the men paying four ſhillings apiece, the women ſixpence apiece, and the children threepence apiece: How many were there of each ſort?*

SOLUTION.

Put x, y and z for the number of men, women and children reſpectively, and conſequently $4x+\frac{y}{2}+\frac{z}{4}$ for the number of ſhillings paid by them; and this laſt condition furniſhes us with the following equation, $4x+\frac{y}{2}+\frac{z}{4}=20$: multiply the whole equation by 4, to take off the fractions, and you will have $16x+2y+z=80$; and the equations will ſtand thus:

Equ. 1ſt, $x+y+z=20$.
2d, $16x+2y+z=80$.

Expunge z by ſubtracting the firſt equation from the ſecond, and you will have $15x+y=60$; whence

Equ. 3d, $y=60-15x$.

Therefore $15x$ muſt be leſs than 60, and x muſt be leſs than 4. Exterminate y by ſubtracting twice the firſt equation from the ſecond, and you will have $14x-z=40$; whence

Equ. 4th, $z=14x-40$.

Therefore $14x$ is greater than 40, and x is greater than 2; therefore x muſt lie between 2 and 4. Now as the values of y and z were expreſſed in the third and fourth equations without fractions, it follows, that whatever whole number is ſubſtituted for x, y and z will come out whole numbers; but they will not come out affirmative, unleſs x be a whole number between 2 and 4; therefore x muſt be 3, and the problem admits of but one ſolution; therefore y or $60-15x$ will be 15, and z or $14x-40$ will be 2: ſo there were three men, fifteen women, and two children.

PROBLEM 14.

219. *Forty one perſons conſiſting of men, women and children at a collation paid forty ſhillings, the men paying four ſhillings apiece, the women three ſhillings apiece, and the children fourpence apiece: How many were there of each ſort?*

 SOLU-

SOLUTION.

Put x, y and z for the number of men, women and children respectively, and consequently $4x+3y+\frac{z}{3}$ for the number of shillings paid by them, and you will have $4x+3y+\frac{z}{3}=40$, or $12x+9y+z=120$; therefore

Equation 1st, $x+y+z=41$.
2d, $12x+9y+z=120$.

Subtract the second equation from twelve times the first, and you will have $3y+11z=372$, and

Equation 3d, $y=124-\frac{11z}{3}$.

Therefore $\frac{11z}{3}$ must be less than 124, and $11z$ less than 372, and z must be less than 34. Subtract nine times the first equation from the second, and you will have $3x-8z=-249$; whence

Equation 4th, $x=\frac{8z}{3}-83$.

Therefore $\frac{8z}{3}$ must be greater than 83, and $8z$ greater than 249, and z must be greater than 31. Hence, and from the third and fourth equations, if x and y be whole numbers and affirmative, z must be some multiple of 3 lying between 31 and 34; but there is only one such multiple, to wit 33; therefore the problem admits but of one solution, and $z=33$; therefore x or $\frac{8z}{3}-83=5$, and y or $124-\frac{11z}{3}=3$: so there were 5 men, 3 women, and 33 children.

PROBLEM. 15.

220. *It is required to divide the number thirty into three such integral parts* x, y *and* z, *that* $2x+9y+15z$ *may make four hundred and nineteen.*

SOLUTION.

Equation 1st, $x+y+z=30$.
2d, $2x+9y+15z=419$.

Subtract twice the first equation from the second, and you will have $7y+13z=359$; whence

Equation 3d, $y=\frac{359-13z}{7}$.

Therefore z must be less than 28, or (which is the same thing) z must not be

be any whole number greater than 27. Subtract the second equation from nine times the first, and you will have $7x - 6z = -149$; therefore

Equation 4th, $x = \frac{6z - 149}{7}$.

Therefore z must not be any whole number less than 25. From the third and fourth equations it appears, that $\frac{359 - 13z}{7}$ and $\frac{6z - 149}{7}$ must both be whole numbers, the possibility whereof might be demonstrated as in the tenth and eleventh problems; but for the sake of variety I shall make it appear thus: $\frac{359 - 13z}{7}$ and $\frac{6z - 149}{7}$ when added together make $\frac{210 - 7z}{7}$, which is reducible to a whole number; therefore if $\frac{6z - 149}{7}$ be a whole number, the other $\frac{359 - 13z}{7}$ must be so too; for otherwise their sum could not be a whole number. To find then the value of z, I argue thus: whatever value of z will make $\frac{6z - 149}{7}$ a whole number, the same will make $\frac{6z - 2}{7}$ a whole number, because $6z - 149$ being divided by 7, there remains $6z - 2$; but 5 is a number which being substituted for z will make $\frac{6z - 2}{7}$ a whole number;

therefore if z be made equal to 5, x and y will both come out whole
numbers, but not both affirmative; for that both may be affirmative, it
is required that z should not be any whole number less than 25, or great-
er than 27: however having found the least value of z that will make
x and y whole numbers, to wit, 5; and finding by either of the equa-
tions $7x - 6z = -149$, or $7y + 13z = 359$ that all the other values
of z must be found by a continual addition of the number 7, I begin
a progression from 5, and carry it on by a continual addition of the num-
ber 7 till I come at a term that lies between the two limits of z (which
will always be possible when the problem is possible), thus: 5, 12, 19,
26: since then the term 26 lies between the two limits of z, to wit,
24 and 28, I conclude that if I make z equal to 26, x and y will both
come out whole numbers and affirmative; therefore $z = 26$, x or $\frac{6z - 149}{7} = 1$, and y or $\frac{359 - 13z}{7} = 3$: so the numbers are 1, 3 and 26; which

upon

upon tryal will anſwer the conditions of the problem, and the problem admits only of this ſolution.

x	y	z.
—17	42	5.
—11	29	12.
— 5	16	19.
1	3	26.
7	—10	33.

PROBLEM 16;

Being a general problem.

221. *To find, if poſſible, three numbers, all integral and affirmative, whoſe ſum is not only given, but alſo the ſum of their products when ſeverally multiplied by three given multiplicators.*

SOLUTION.

Let x, y and z be the three numbers ſought, putting x for that number whoſe multiplicator is the greateſt, and z for that whoſe multiplicator is the leaſt, or negative if any of them be negative, or if two of them happen to be negative, put z for that number whoſe multiplicator is the greater negative; then putting y for the middle term, you will have two equations, one whoſe abſolute term is the given ſum of the three numbers ſought, and another whoſe abſolute term is the given ſum of the products. By the help of theſe two equations expunge z thus: multiply the firſt equation by the coefficient of z in the ſecond, and then ſubtracting that product from the ſecond equation, you will fall into a third of this form, $Ax+By=C$: divide this whole equation by the greateſt common meaſure of A and B, and let the reſult be $ax+by=c$; then from this equation, and from the manner of deriving it, I ſhall deduce the following obſervations.

1ſt. If the directions given for the notation be duly obſerved, the numbers a and b will always be integral, affirmative, and prime to each other, and a will always be greater than b.

2d. If the numbers x and y be integral and affirmative, the number $\overline{ax+by}$ or c muſt be ſo too; though it does not follow *e converſo*, that if c be integral and affirmative, x and y muſt both be affirmative.

3d. If any one value of y be known, the correſpondent values of x and z will eaſily be had by making $x=\frac{c-by}{a}$ and $z=s-x-y$, putting

s for

s for the given ſum of the three numbers ſought: whence it follows, that if x and y be whole numbers, z muſt neceſſarily be ſo too.

4th. If the leaſt value of y be known, and the problem admits of more ſolutions than one, as many other ſucceſſive values of y as we ſhall have occaſion for may eaſily be obtained by a continual addition of the number a; whence the correſpondent values of x will form a decreaſing arithmetical progreſſion whoſe common difference is b, and thoſe of z another decreaſing arithmetical progreſſion whoſe common difference is $\overline{a-b}$. The two firſt parts of my aſſertion are evident from art. 213, and the laſt I thus demonſtrate: If the values of y increaſe by a and thoſe of x decreaſe by b, the ſeveral values of the ſum $\overline{x+y}$ conſidered as one number will increaſe by $\overline{a-b}$; but as the ſum $\overline{x+y}$ increaſes, the third quantity z muſt be diminiſhed, becauſe in the ſame problem the whole ſum $\overline{x+y+z}$ is always the ſame; therefore the ſucceſſive values of z will form a decreaſing arithmetical progreſſion whoſe common difference is $\overline{a-b}$.

5th. From this laſt obſervation it follows, that the raiſing of y is the ſinking of x and z, and *vice verſa*; inſomuch that when y is the leaſt in it's kind, x and z will be the greateſt in their's to ſolve the problem; and if in this ſtate both be affirmative, they will do it; but if either or both happen to be negative, the problem will be impoſſible, and can have no ſolution.

Being thus prepared, I now proceed to find two ſuch whole numbers as, being ſubſtituted for x and y, will make $ax+by=c$; which may be effected in the manner following: by the quotients of a continual diviſion from a and b, form a ſet of equations according to art. 175; and as the original numbers a and b are prime to each other, the abſolute term of the two laſt equations will be ± 1; take that equation wherein -1 is concerned, that is, take the laſt equation that ſtands in an even place, and it will be of this form, $ae-bd=-1$; change the ſigns, and you will have $-ae+bd=+1$; multiply all by c, and you will have $-ace+bcd=c$: make $-ce=x$ and $+cd=y$, and you will have two whole numbers x and y with this property, that $ax+by$ will be equal to c: but here as x is negative, it will be proper to try to raiſe it (if poſſible) to an affirmative number by ſinking the value of y as low as we can, that is, by throwing away the number a from cd as often as we can, that is, by dividing cd by a; for if this be done, the remainder if there be any, or the diviſor a if there be none, will be the leaſt value of y; and if by this means x and z come out affirmative, the problem will admit of one or more ſolutions in affirmative numbers; but if x ſtill happen to be negative, or if z comes out negative, there is

no

no raiſing theſe negative values without ſinking the value of y, which is already as low as it can be, and therefore in this caſe the problem can have no affirmative ſolution.

As for the number d, ſince that is the coefficient of b in the laſt equation that ſtands in an even place, and ſince theſe coefficients are nothing elſe but the terms of a ſeries begun from 0 and 1, and continued by the quotients of a continual diviſion made from a and b, it is evident that if ſuch a diviſion be actually made, and the number of quotients be even, an unit muſt be deducted from the laſt, for a reaſon given in art. 175; but but if the number of quotients be odd, the laſt muſt be thrown away, as leading to a wrong equation; and then if a ſeries from 0 and 1 be computed by the reſt of the quotients, the laſt term of the ſeries will be d; therefore in practice the number d may be computed by the help of this ſeries only, without medling with the other parts of the equations wherein theſe terms are concerned.

EXAMPLE I.

Let the equations be

1ſt, $x+y+z=s$.

2d, $10x+5y+2z=r$.

Then ſubtracting twice the firſt equation from the ſecond, you will have $8x+3y=r-2s$; and becauſe the numbers 8 and 3 are prime to each other, the equation can be reduced no lower; therefore $a=8$, $b=3$, $c=r-2s$, and $d=3$; for the quotients of a continual diviſion from 8 and 3 will be 2, 1, 2; whence, dropping the laſt becauſe their number is odd, with the remaining quotients 2 and 1 form the ſeries 0, 1, 2, 3, and you will have $d=3$, and $cd=3c$; thus you will have a particular canon adapted to the multiplicators 10, 5 and 2, as follows: make $r-2s=c$; divide $3c$ by 8, and the remainder will be y; whence $x=\frac{c-3y}{8}$, and $z=s-x-y$. As for inſtance, let $s=20$ and $r=53$, that is, let it be required to divide the number 20 into three ſuch parts, that ten times the firſt part, five times the ſecond, and twice the third may all together make 53: here $r-2s$ or $c=13$, and $3c=39$, which being divided by 8 leaves 7 for a remainder; therefore in this caſe $y=7$, $\frac{13-3y}{8}$ or $x=-1$, and $s-x-y$ or $z=14$: ſo the numbers are -1, $+7$ and $+14$; therefore this problem cannot be reſolved in affirmative numbers, but in others it may; for $-1\times+10$, $+7\times5$, $+14\times2$, that is, $-10+35+28=53$. Again, ſuppoſing the multiplicators 10, 5 and 2 to continue, and making $s=20$ as before, let now $r=107$, and you will have $c=67$, $3c=201$, $y=1$, $x=8$, and $z=11$: ſo the

numbers

numbers are 8, 1 and 11, which will ſolve the problem; for $8 \times 10 + 1 \times 5 + 11 \times 2$, that is, $80 + 5 + 22 = 107$. But this problem will alſo admit of two more affirmative ſolutions, which according to the fourth obſervation are 5, 9 and 6; 2, 17 and 1.

EXAMPLE 2.

Let it be required to divide a given number as s into three ſuch parts, that three times one part, a third part of the ſecond, and a fifth part of the laſt may again make up the number s.

Here the equations are $x + y + z = s$, and $3x + \frac{y}{3} + \frac{z}{5} = s$; but to fit this laſt equation for uſe, I reduce it to integral terms by multiplying the whole equation by 15, and ſo have $45x + 5y + 3z = 15s$; therefore the equations when fitted for uſe are

$$\text{1ſt, } x + y + z = s, \text{ and}$$
$$\text{2d, } 45x + 5y + 3z = 15s.$$

Subtract three times the firſt equation from the ſecond, and you will have $42x + 2y = 12s$; divide the whole equation by 2, becauſe 2 is the greateſt common meaſure of 42 and 2, and you will have $21x + 1y = 6s$. Here then $a = 21$, $b = 1$, $c = 6s$, and $d = 1$; for the ſingle quotient of a divided by b, or of 21 divided by 1 being rejected for it's ſingularity, there will be no remaining quotient for continuing the ſeries from 0 and 1; therefore in this caſe 1 the latter term of the ſeries muſt be taken for d; whence cd the dividend will be $6s$, and we ſhall have the following canon particularly adapted to this problem; I mean, where the multiplicators are 3, $\frac{1}{3}$ and $\frac{1}{5}$, and where the ſum of the products is to be the ſame with the ſum of the numbers: divide $6s$ by 21, and the remainder will be y; whence we ſhall have $x = \frac{6s - y}{21}$, and $z = s - x - y$. As for inſtance, let it be required to divide 100 into three ſuch parts, that three times the firſt part, a third part of the ſecond, and a fifth part of the third may all together make the ſame number 100: here $6s = 600$, which being divided by 21 leaves 12 for y; whence $x = 28$, and $z = 60$: ſo the numbers 28, 12 and 60 will ſolve the problem; for $28 \times 3 + \frac{12}{3} + \frac{60}{5}$, that is, $84 + 4 + 12 = 100$. The fourth obſervation finds alſo two other affirmative ſolutions of this problem, to wit, 27, 33, 40; and 26, 54, 20; and theſe are all; for another operation would ſink z to nothing.

EXAMPLE 3.

Let the ſum of the products be ſtill equal to the ſum of the numbers, and let the multiplicators be 3, 1 and $-\frac{1}{3}$, that is, let $3x + y - \frac{z}{3} = s$, or $9x + 3y - z = 3s$, and the equations will now be

1st, $x+y+z=s$.
2d, $9x+3y-z=3s$.

Multiply the first equation by -1, and then subtract it from the second, or (which is the same in effect) add the first equation to the second, and you will have $10x+4y=4s$; divide the whole by 2 (the greatest common measure of 10 and 4) and you will have $5x+2y=2s$; therefore in this case $a=5$, $b=2$, and $c=2s$. Now the quotients of a continual division from 5 and 2 are 2 and 2; call them 2 and 1, and with them compute the series 0, 1, 2, 3, and you will have $d=3$, and the canon for the multiplicators 3, 1 and $-\frac{1}{3}$, where the sum of the products is to be equal to the sum of the numbers, is as follows: divide $6s$ by 5, and the remainder will be y; then make $\frac{2s-2y}{5}=x$, and $s-x-y=z$. As for example, let $s=20$; then will $6s=120$, which divided by 5 will have nothing or 5 for a remainder; therefore in this case $y=5$, $x=6$, and $z=9$, and the numbers in order are 6, 5 and 9, which will answer the conditions; for first $6+5+9=20$, and secondly $18+5-3=20$. But according to the fourth observation, this problem will admit of two more solutions, which are 4, 10 and 6; and 2, 15 and 3.

N. B. 1st. In the abovewritten canon we made y the remainder of $6s$ divided by 5; but we might have thrown away $5s$, and have made y the remainder of $1s$ divided by 5; and the same reduction may be made in all other cases where the coefficient of s exceeds the divisor.

2dly. This problem includes the 9th, 10th, 11th, 12th, 13th, 14th and 15th, with an infinite number of other particular ones of this kind: but if this solution be applied to them by way of examples, care must be taken to change their notation, and to fix it according to the directions given at the beginning of this solution.

Of the magic square.

PROBLEM 17.

222. *Let any odd square number as 49 be proposed whose square root is 7; and let any square figure be divided into 49 lesser square cells, to wit, into 7 ranks of cells, and every rank into 7 cells: It is required to distribute among these cells all the natural numbers from 1 to 49 inclusively, so that the sum of all the numbers in every row, whether taken horizontally, perpendicularly or diagonally may be the same; which figure thus constructed is commonly called a magic square.*

SOLU-

SOLUTION.

As 7 is the ſide of the ſquare propoſed, let the ſeven firſt digits 1, 2, 3, 4, 5, 6, 7 be repreſented by the ſeven firſt letters of the alphabet *a*, *b*, *c*, *d*, *e*, *f*, *g* reſpectively; and let theſe letters be ſeven times repeated in the ſame order in ſeven diſtinct rows, placed exactly one under another; and let the difference of the rows conſiſt only in this, that every inferior row is to begin with the ſecond term next above it; and they will ſtand as in the firſt figure; where the firſt or uppermoſt row is *a*, *b*, *c*, *d*, *e* *f*, *g*; the ſecond *b*, *c*, *d*, *e*, *f*, *g*, *a*; the third *c*, *d*, *e*, *f*, *g*, *a*, *b*; &c. This ſcheme, which is in form of a ſquare, is called the primitive ſquare, becauſe upon it is founded the conſtruction of the magic ſquare next to be conſidered.

Figure 1.

a	*b*	*c*	*d*	*e*	*f*	*g*
b	*c*	*d*	*e*	*f*	*g*	*a*
c	*d*	*e*	*f*	*g*	*a*	*b*
d	*e*	*f*	*g*	*a*	*b*	*c*
e	*f*	*g*	*a*	*b*	*c*	*d*
f	*g*	*a*	*b*	*c*	*d*	*e*
g	*a*	*b*	*c*	*d*	*e*	*f*

Let now any other ſquare divided into fortynine cells be prepared, as in the ſecond figure, which I ſhall by way of anticipation call the magic ſquare; and let the blank cells hereof be filled up by the help of the primitive ſquare, in the manner following.

Figure 2.

$7d+d$	$7e+e$	$7f+f$	$7g+g-49$	$7a+a$	$7b+b$	$7c+c$
$7e+c$	$7f+d$	$7g+e-49$	$7a+f$	$7b+g$	$7c+a$	$7d+b$
$7f+b$	$7g+c-49$	$7a+d$	$7b+e$	$7c+f$	$7d+g$	$7e+a$
$7g+a-49$	$7a+b$	$7b+c$	$7c+d$	$7d+e$	$7e+f$	$7f+g$
$7a+g$	$7b+a$	$7c+b$	$7d+c$	$7e+d$	$7f+e$	$7g+f-49$
$7b+f$	$7c+g$	$7d+a$	$7e+b$	$7f+c$	$7g+d-49$	$7a+e$
$7c+e$	$7d+f$	$7e+g$	$7f+a$	$7g+b-49$	$7a+c$	$7b+d$

Firſt, beginning with the firſt horizontal row of the primitive ſquare, let all the numbers in this row be ſucceſſively multiplied by 7 the ſide of the propoſed ſquare, proceeding from left to right; and to the ſeveral products of this multiplication add in order the numbers of the ſeventh and loweſt horizontal row, and you will have the numbers $\overline{7a+g}$, $\overline{7b+a}$, $\overline{7c+b}$, $\overline{7d+c}$, $\overline{7e+d}$, $\overline{7f+e}$, $\overline{7g+f}-40$, with which numbers the next horizontal row below the middle one of the magic ſquare is to be filled from left to right, which in the preſent caſe happens to be the fifth row, reckoning from the uppermoſt: but here (as you may ſee in the figure) 49 muſt be thrown away from every greater number; for ſhould we receive any number greater than 49 into this ſquare, places would be wanting for others that are leſs than 49, which the nature of the problem will not allow of.

2dly. Having furniſhed this row of the magic ſquare, deſcend all along with your multiplication, and aſcend with your addition in the primitive ſquare, ſo that the row to be multiplied, and the row to be added may always be equally diſtant from the middle row on each ſide; that is, multiply all the numbers in the ſecond row of the primitive ſquare by 7, and add to every product a correſpondent number in the ſixth row, and you will have the numbers $\overline{7b+f}$, $\overline{7c+g}$, $\overline{7d+a}$, &c, for furniſhing the ſixth row of the magic ſquare.

3dly. In like manner, ſeven times the third row together with the fifth, will furniſh the ſeventh row of the magic ſquare.

4thly. We come now to the middle row of the primitive ſquare, wherein both the row to be multiplied, and the row to be added meet; therefore this row will of itſelf alone, without the aſſiſtance of any other, furniſh out numbers for the firſt row of the magic ſquare, to wit, $\overline{7d+d}$, $\overline{7e+e}$, $\overline{7f+f}$, &c.

Figure 1.

a	b	c	d	e	f	g
b	c	d	e	f	g	a
c	d	e	f	g	a	b
d	e	f	g	a	b	c
e	f	g	a	b	c	d
f	g	a	b	c	d	e
g	a	b	c	d	e	f

5thly. In like manner ſeven times the fifth row of the primitive ſquare together with the third, ſeven times the ſixth together with the ſecond, and ſeven times the loweſt together with the higheſt, will furniſh numbers for the ſecond, third and fourth horizontal rows of the magic ſquare. And thus the whole figure will be furniſhed, at leaſt in letters: my next taſk therefore muſt be to demonſtrate that this figure has all the properties of a magic ſquare.

Figure 2.

7d+d	7e+e	7f+f	7g+g—49	7a+a	7b+b	7c+c
7e+c	7f+d	7g+e—49	7a+f	7b+g	7c+a	7d+b
7f+b	7g+c—49	7a+d	7b+e	7c+f	7d+g	7e+a
7g+a—49	7a+b	7b+c	7c+d	7d+e	7e+f	7f+g
7a+g	7b+a	7c+b	7d+c	7e+d	7f+e	7g+f—49
7b+f	7c+g	7d+a	7e+b	7f+c	7g+d—49	7a+e
7c+e	7d+f	7e+g	7f+a	7g+b—49	7a+c	7b+d

And firſt I will demonſtrate that it contains all the natural numbers from 1 to 49 incluſively: for whoſoever traces the number 7g through all the horizontal rows, will find $\overline{7g-49}$ combined with all the letters a, b, c, d, e, f, g, though in another order than as they are here placed; which

which numbers (ſince $\overline{7g-49}=0$) are $a, b, c, d, e, f, g,$ that is, all the numbers from 1 to 7 incluſively: in tracing $7a$ after the ſame manner he will find, when reduced to order, the numbers $\overline{7a+a}$, $\overline{7a+b}$, $\overline{7a+c}$, $\overline{7a+d}$, $\overline{7a+e}$, $\overline{7a+f}$, $\overline{7a+g}$, which ſeries includes all numbers from 8 to 14: he will likewiſe find all the terms of the ſeries $\overline{7b+a}$, $\overline{7b+b}$, $\overline{7b+c}$, $\overline{7b+d}$, $\overline{7b+e}$, $\overline{7b+f}$, $\overline{7b+g}$, which takes in all the numbers from 15 to 21: and ſo on to $\overline{7f+g}=49$. In the next place, it is plain that the ſum of all the horizontal and perpendicular rows will be the ſame: for whoever conſiders theſe rows with any attention, cannot but ſee, that the ſame letters are multiplied, and the ſame added in every one of theſe rows, only in a different order; and therefore the ſums muſt all be the ſame. We come in the laſt place to conſider the two diagonal rows: now if we examine that which deſcends from left to right, from $\overline{7d+d}$ to $\overline{7b+d}$, we ſhall find the numbers of this firſt diagonal row to be $\overline{7d+d}$, $\overline{7f+d}$, $\overline{7a+d}$, $\overline{7c+d}$, $\overline{7e+d}$, $\overline{7g+d-49}$, $\overline{7b+d}$; but $\overline{7d+7f+7a+7c+7e+7g+7b}=\overline{7a+7b+7c+7d+7e+7f+7g}$; therefore the ſum of the firſt diagonal row, which is not affected by changing the order of the numbers, will be the ſame with the ſum of theſe numbers $\overline{7a+d}$, $\overline{7b+d}$, $\overline{7c+d}$, $\overline{7d+d}$, $\overline{7e+d}$, $\overline{7f+d}$, $\overline{7g+d-49}$; but d is the middle term of the progreſſion $a, b, c, d, e, f, g,$ and therefore the ſum of this progreſſion will be the ſame as if all the terms were d's, that is, the ſum of this progreſſion will be $7d$; if therefore inſtead of the addititious terms in the numbers abovewritten, which are d ſeven times repeated, we add the terms of the progreſſion $a, b, c, d, e, f, g,$ which will not alter the ſum of the whole, we ſhall have the ſum of the firſt diagonal row equivalent to the ſum of the numbers $\overline{7a+a}$, $\overline{7b+b}$, $\overline{7c+c}$, $\overline{7d+d}$, $\overline{7e+e}$, $\overline{7f+f}$, $\overline{7g+g-49}$, which is the ſame with the ſum of the firſt horizontal row, and conſequently of every other row: in the other diagonal row which aſcends from left to right, from $\overline{7c+e}$ to $\overline{7c+c}$, we find the number $7c$ ſeven times repeated, and ſeverally connected with all the letters $a, b, c, d, e, f, g,$ though in a different order; but $c=d-1$; therefore $7c=7d-7=\overline{a+b+c+d+e+f+g-7}$; therefore $7c$ ſeven times repeated equals $\overline{7a+7b+7c+7d+7e+7f+7g-49}$; therefore the ſum of this diagonal row will be the ſame as if the order and value of it's terms were as follows, to wit, $\overline{7a+a}$, $\overline{7b+b}$, $\overline{7c+c}$, $\overline{7d+d}$, $\overline{7e+e}$, $\overline{7f+f}$, $\overline{7g+g-49}$, that is, the ſum of this diagonal row is the ſame with that of all the reſt. *Q. E. D.*

Hitherto

Hitherto we have filled our magic ſquare with letters, in order to give a clearer idea of it's nature and compoſition: but if we would fill our ſquare with numbers, as the problem requires, we muſt firſt make a primitive ſquare in numbers, as in figure 3; where the firſt or uppermoſt row is 1, 2, 3, 4, 5, 6, 7; the ſecond 2, 3, 4, 5, 6, 7, 1; the third 3, 4, 5, 6, 7, 1, 2; &c: and if by the help of this, a magic ſquare be conſtructed according to the directions before given, we ſhall have ſuch a one as is exhibited in the fourth figure.

Figure 3.

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

Figure 4.

32	40	48	7	8	16	24
38	46	5	13	21	22	30
44	3	11	19	27	35	36
1	9	17	25	33	41	49
14	15	23	31	39	47	6
20	28	29	37	45	4	12
26	34	42	43	2	10	18

Whoever duly conſiders the compoſition of the magic ſquare, and the diſpoſition of it's numbers, will be able to draw (or at leaſt to ſee the evidence of) the following conſectaries.

Firſt, that whatever be the ſide of the ſquare, unity will always ſtand in the firſt cell of the middle horizontal row on the left hand: for $7g + a - 49 = 1$; and if we make 7 or g (the ſide of the ſquare) equal to r, we ſhall always have for that cell $rr + a - rr = 1$.

2d. If the numbers 1, 2, 3, 4, 5, 6, 7 in the primitive ſquare had been continued *ad infinitum*, the numbers in every horizontal row of the magic ſquare, except where 49 is to be caſt away, would have increaſed from left to right in an arithmetical progreſſion, whoſe common difference is 8 $(r + 1)$: for in the firſt row for inſtance, (and the reaſon is the ſame for all the reſt,) the products $7d$, $7e$, $7f$, $7g$, &c make an increaſing arithmetical progreſſion whoſe common difference is 7; and the parts to be added, *viz.* d, e, f, g, &c make another increaſing progreſſion whoſe difference is 1; therefore both together, to wit, $7d + d$, $7e + e$, $7f + f$, $7g + g$ &c, will make a progreſſion whoſe common difference is 8.

3d. But becauſe the numbers in the primitive ſquare are not continued *ad infinitum*, but always break off after 7, returning back again to 1, that is, from g to a, this occaſions an irregular term wherever the number a is to be added: I call it ſo, becauſe it's exceſs above the term that is before it in the ſame row will be but unity, whereas according to the law

law of this progreſſion, it ought to be 8. Of theſe irregular terms there will be but one in every horizontal row; they are marked in the fourth figure, and beginning with unity are as follows; 1, 15, 29, 43, 8, 22, 36, forming an arithmetical progreſſion whoſe common difference is 14 ($2r$).

4th. From what has been laid down in the three foregoing conſectaries, and particularly from a due obſervation of the places of the irregular terms, being taken in the ſame order as they were given by us in the laſt, we have a new way of conſtructing a magic ſquare more expeditious than the former, as being effected without the help of a primitive ſquare, thus: Form from unity an arithmetical progreſſion whoſe common difference is $2r$, and the number of terms r, which progreſſion in the preſent caſe will be 1, 15, 29, 43, 8, 22, 36; of this progreſſion place the firſt term in the firſt cell of the middle horizontal row on the left hand of the magic ſquare to be conſtructed; then deſcending from thence diagonally to the right, fill the cells you paſs through with as many of the other terms of the foregoing progreſſion as you can: theſe terms in the preſent caſe are 15, 29, 43; but if this motion be further continued, the next term 8 will fall without the ſquare under the fifth perpendicular row from the left hand, as you ſee in the figure: transfer it therefore to the uppermoſt cell of the ſame fifth perpendicular row, and then moving downwards diagonally as before, you will find room for the remaining terms of the progreſſion, to wit, 22 and 36. The irregular terms being thus placed in their proper cells, all the other cells will be eaſily filled thus: ſuppoſe I would fill the firſt horizontal row; I look for the irregular term already placed there, and find it to be 8; from this term 8 I form an arithmetical progreſſion whoſe common difference is 8, and then moving in the ſame row towards the right hand, I inſert as many of the other terms of this progreſſion as I can, which terms in the preſent caſe are 16 and 24; then finding I can move no further this way, I continue the reſt of the progreſſion from the other end of the ſame horizontal row, writing down, from left to right, the numbers 32, 40, 48 and 7: thus the firſt row will be full; and the reſt muſt be furniſhed by a like proceſs.

5th. Since in a magic ſquare, the ſum of every row, which way ſoever taken, is the ſame; if any one has a mind to eſtimate this ſum without actually conſtructing the figure, he may reaſon thus: let 7 be the ſide of the ſquare; then it is plain that in this ſquare will be contained all the natural numbers from 1 to 49 incluſively, which in their natural order make an arithmetical progreſſion, whereof the leaſt term is 1, the greateſt 49, and the number of terms 49: add the greateſt and leaſt terms together, and the ſum will be 50, whoſe half is 25; therefore 25 is an arithmetic mean between the extremes 1 and 49; and 49 times 25 will be the ſum of the whole progreſſion by art.

124; therefore the ſum of all the numbers in the whole ſquare, or the ſum of the numbers in all the ſeven horizontal, or in all the ſeven perpendicular rows, is 49 times 25; therefore the ſum of the numbers in any one row is 7 times 25 or 175: and the reaſoning is the ſame in any other caſe where the ſide of the ſquare is an odd number. The ſum of all the numbers in any one row is $\frac{r^2+1}{2}\times r$; whence I infer, that if $\frac{r^2+1}{2}$ be made equal to m, that is, if m be an arithmetic mean between the extremes 1 and r^2, the ſum of every row will be the ſame as if it's cells were all filled with the ſame number m.

There are many other ways of conſtructing magic ſquares both odd and even, as alſo many other ſurprizing properties thereto relating, all which I here paſs by, becauſe, though this ſpeculation be in itſelf very curious and entertaining, yet it cannot be denied but that it is of very little or no uſe in any other parts of the Mathematics. Whoever would ſee more of theſe matters, may conſult the Memoirs of the Royal Academy of Sciences for the years 1705 and 1710, where he will find this ſubject almoſt exhauſted by the learned and juſtly celebrated Mathematicians M. *De La Hire* and M. *Sauveur*.

